Chapter 14

Multiple integrals

14.4 Double integral in polar coordinate form

We are given a region D by

$$D = \{(r, \theta) \mid \phi_1(\theta) \le r \le \phi_2(\theta), \quad \alpha \le \theta \le \beta\}.$$

We divide D by the curves $\theta =$ constant and the lines $\Delta\theta = (\beta - \alpha)/l$,

$$r_0 = \Delta r,\ r_1 = 2\Delta r, \ldots, r_{m+1} = m\Delta r,$$

and

$$\theta_0 = \alpha,\ \theta_1 = \alpha + \Delta\theta,\ \ldots,\ \theta_{l+1} = \alpha + l\Delta\theta = \beta.$$

Choose any point (r_k, θ_k) in ΔA_k and consider the Riemann sum

$$\mathcal{R}(f, n) = S_n = \sum_{k=1}^{n} f(r_k, \theta_k)\Delta A_k.$$

Let $\delta = \max_{i,j}\{\Delta r_i, \Delta\theta_j\}$. If the limit $\lim_{n\to\infty} \mathcal{R}(f, n)$ exists (as δ approaches 0), then it is defined as the integral of f on D and we write

$$\iint_D f(r, \theta)\, dA.$$

Assume the point (r_k, θ_k) is at the center of ΔA_k(figure **??**, left). The area

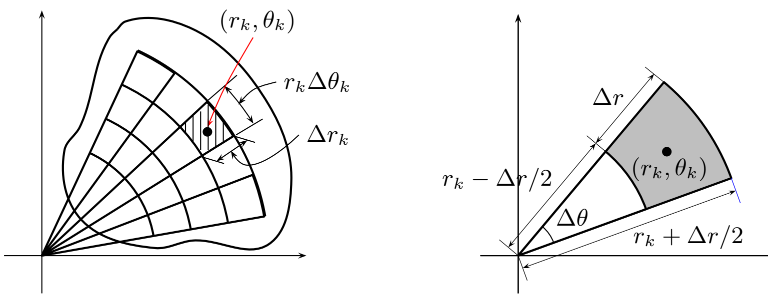


Figure 14.1: Partition in polar coordinate

of ΔA_k is

$$\frac{1}{2}\left(r_k + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r_k - \frac{\Delta r}{2}\right)^2 \Delta\theta = r_k \Delta r \Delta\theta.$$

Proposition 14.4.1. *If D is given by $D = \{(r,\theta) \mid \phi_1(\theta) \le r \le \phi_2(\theta), \alpha \le \theta \le \beta\}$, the integral of f can be evaluated as the iterated integral:*

$$\iint_D f(r,\theta)\, dA = \int_\alpha^\beta \int_{\phi_1(\theta)}^{\phi_2(\theta)} f(r,\theta) r\, dr d\theta.$$

Example 14.4.2. Find the area of the region inside the cardioid $r = 1 - \sin\theta$.

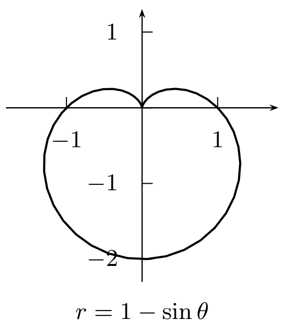


Figure 14.2: $r = 1 - \sin\theta$

sol. cardioid. We see $0 \le r \le 1 - \sin\theta$

$$\begin{aligned}\int_0^{2\pi} \int_{r=0}^{r=1-\sin\theta} r\, dr d\theta &= \int_0^{2\pi} \left[\frac{r^2}{2}\right]_{r=0}^{r=1-\sin\theta} d\theta \\ &= \int_0^{2\pi} \frac{(1-\sin\theta)^2}{2}\, d\theta\end{aligned}$$

$$= \frac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta + \sin^2\theta)\, d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} (1 - 2\sin\theta + \frac{1-\cos 2\theta}{2})\, d\theta$$
$$= \frac{1}{2}\left[\theta + 2\cos\theta + \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{2\pi}$$
$$= \frac{3}{2}\pi.$$

□

□

Example 14.4.3. The area inside of the cardioid $r = 1 + \cos\theta$ and outside of the unit circle $r = 1$.

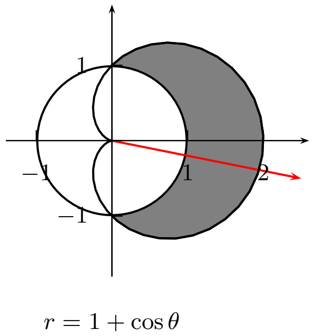


Figure 14.3: Find the limits of integral $r = 1$, $r = 1 + \cos\theta$

Example 14.4.4. Change the integral $\iint f(x, y)\, dxdy$ to polar coordinate.

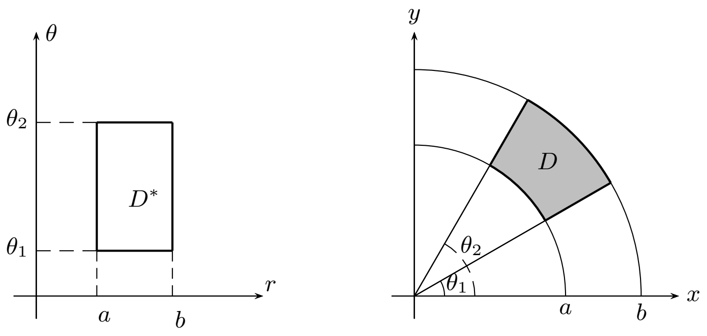
sol. Since $x = r\cos\theta$, $y = r\sin\theta$, we can let $T(r, \theta) = (r\cos\theta, r\sin\theta)$. Then Jacobian is

$$\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Hence

$$\iint f(x, y)\, dxdy = \iint f(r\cos\theta, r\sin\theta)\, r dr d\theta.$$

□



Example 14.4.5. D is between two concentric circles: $x^2+y^2=4$, $x^2+y^2=1(x,y\geq 0)$. Find the integral

$$\iint_D \sqrt{x^2+y^2+1}\,dxdy.$$

Here D is the quoter of the annulus $\sqrt{1-x^2}\leq y\leq\sqrt{4-x^2}$.

sol. Use polar coordinate. We see the domain of integration in (r,θ) is

$$D^*=\{(r,\theta)|1\leq r\leq 2, 0\leq\theta\leq\pi/2\}.$$

$$\begin{aligned}\iint_D \sqrt{x^2+y^2+1}\,dxdy &= \iint_{D^*}\sqrt{r^2+1}r\,drd\theta\\ &= \int_0^{\pi/2}\int_1^2 \frac{1}{2}\sqrt{r^2+1}(2r)drd\theta\\ &= \int_0^{\pi/2}\frac{1}{3}(r^2+1)^{3/2}|_1^2 d\theta\\ &= \int_0^{\pi/2}\frac{1}{3}(5^{3/2}-2^{3/2})d\theta=\frac{\pi}{6}(5^{3/2}-2^{3/2}).\end{aligned}$$

□

Example 14.4.6 (The Gaussian integral)**.** Show that

$$\int_{-\infty}^{\infty} e^{-x^2}dx=\sqrt{\pi}.$$

To compute this, let us first observe

$$\begin{aligned}\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 &= \int_{-\infty}^{\infty} e^{-x^2}dx \int_{-\infty}^{\infty} e^{-y^2}dy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dxdy \\ &= \lim_{a\to\infty}\iint_{D_a} e^{-(x^2+y^2)}dxdy.\end{aligned}$$

Thus it is necessary to compute

$$\iint_{D_a} e^{-(x^2+y^2)}dxdy.$$

By

$$\begin{aligned}\iint_{D_a} e^{-(x^2+y^2)}dxdy &= \int_0^{2\pi}\int_0^a e^{-r^2}r\,drd\theta = \int_0^{2\pi}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_0^a \\ &= -\frac{1}{2}\int_0^{2\pi}(e^{-a^2}-1)d\theta = \pi(1-e^{-a^2}).\end{aligned}$$

Let $a \to \infty$. Then we obtain the result.

14.5 Triple integrals in rectangular coordinates

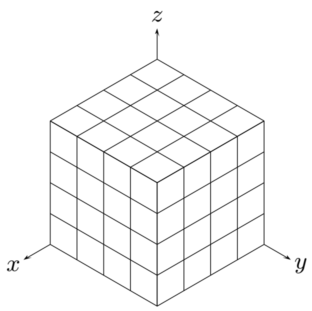


Figure 14.4: partition of box

Definition 14.5.1. Assume $D = [a,b] \times [c,d] \times [p,q]$ be a box. Then we subdivide intervals $[a,b]$, $[c,d]$ and $[p,q]$ into n -intervals

$$a = x_0 < x_1 < \cdots < x_n = b,$$
$$c = y_0 < y_1 < \cdots < y_n = d,$$
$$p = z_0 < z_1 < \cdots < z_n = q,$$

and call the resulting subboxes $D_{jk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k]$ a **partition** of D.

Definition 14.5.2. We let $\Delta V_{ijk} = \Delta x_i \Delta y_j \Delta z_k$ $(i, j, k = 1, \dots, n)$ Then the Riemann sum becomes

$$\mathcal{R}(f, n) = S_n = \sum_{i,j,k=1}^{n} f(c_{ijk}) \Delta V_{ijk}.$$

Here c_{ijk} is any point in the subbox D_{ijk}.

Definition 14.5.3. If $\lim_n S_n = S$ exists independently of the choice of c_{ijk}, then we say f is integrable in D and call S the **triple integral** and we write

$$\iiint_D f dV, \quad \iiint_D f(x, y, z) dV, \text{ or } \quad \iiint_D f(x, y, z) dxdydz.$$

Reduction to iterated integral

Theorem 14.5.4 (Fubini's theorem)**.** *Suppose f is continuous on $D = [a, b] \times [c, d] \times [p, q]$. The triple integral $\iiint_D f(x, y, z)dxdydz$ equals with any of the following integrals.*

$$\int_p^q \int_c^d \int_a^b f(x, y, z)\, dxdydz, \quad \int_p^q \int_a^b \int_c^d f(x, y, z)\, dydxdz, \text{ etc.}$$

Elementary regions

Suppose $R = \{(x, y) \mid \phi_1(x) \le y \le \phi_2(x), \quad a \le x \le b\}$ is an elementary region in xy-plane and there are continuous functions $\gamma_1(x, y)$, $\gamma_2(x, y)$ such that

$$D = \{(x, y, z) \mid \gamma_1(x, y) \le z \le \gamma_2(x, y), \quad (x, y) \in R\}. \tag{14.1}$$

Then D is called an **elementary region of type 1**.

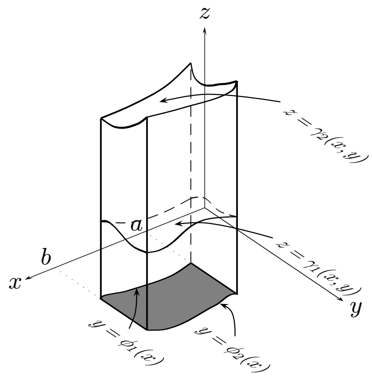


Figure 14.5: elementary region of type 1

Integrals over elementary regions

Then the integral on an elementary region D given above is computed by

$$\begin{aligned}\iiint_D f\,dV &= \iint_R \int f(x,y,z)\,dzdA\\ &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \int_{\gamma_1(x,y)}^{\gamma_2(x,y)} f(x,y,z)\,dzdydx.\end{aligned}$$

Example 14.5.5. Find the volume of radius 1.

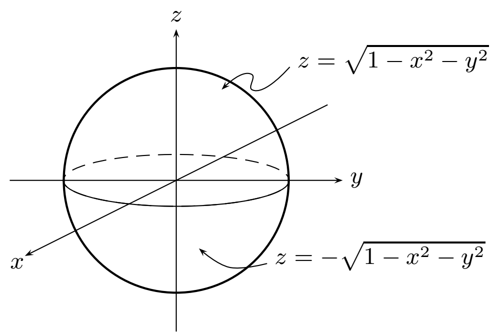


Figure 14.6: $x^2 + y^2 + z^2 = 1$

sol. Unit ball is described by $x^2 + y^2 + z^2 \le 1$. The volume is (Figure **??**)

$$\int_D 1\,dV, \quad D = \{(x,y,z) \mid x^2 + y^2 + z^2 \le 1\}.$$

Here we can take $R = \{(x,y) \mid x^2 + y^2 \le 1\}$ and $D = \{-\sqrt{1 - x^2 - y^2} \le z \le$

$\sqrt{1-x^2-y^2}, (x,y) \in R\}$. Hence

$$\begin{aligned}\iint_R \int dzdydx &= \iint_R \int_{z=-\sqrt{1-x^2-y^2}}^{z=\sqrt{1-x^2-y^2}} 1\, dzdydx \\ &= 2\int_R \sqrt{1-x^2-y^2}\, dydx \\ &= 2\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2-y^2}\, dydx.\end{aligned}$$

This integral can be computed by letting $\sqrt{1-x^2} = a$

□

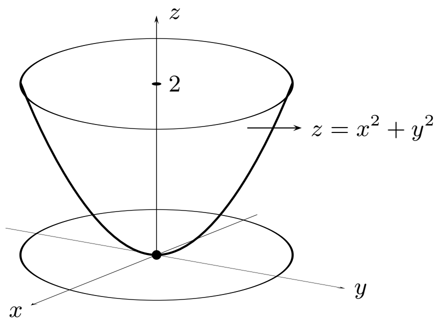


Figure 14.7: $z = x^2 + y^2$, $z = 2$

Example 14.5.6. Let W be bounded by $x = 0, y = 0, z = 2$ and the surface $z = x^2 + y^2$ where $x \geq 0, y \geq 0$. Find $\iiint_W x\, dxdydz$.

sol. Method1. We describe the region by type 1.

$$0 \leq x \leq \sqrt{2}, \quad 0 \leq y \leq \sqrt{2-x^2}, \quad x^2 + y^2 \leq z \leq 2.$$

$$\begin{aligned}\iiint_W x\, dxdydz &= \int_0^{\sqrt{2}} \left[\int_0^{\sqrt{2-x^2}} \left(\int_{x^2+y^2}^{2} x\, dz\right)dy\right] dx \\ &= \frac{8\sqrt{2}}{15}.\end{aligned}$$

□

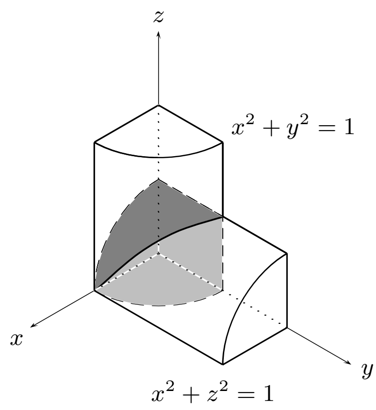


Figure 14.8: common region of two cylinders

Example 14.5.7 (Example 1 p.911)**.** Find the volume of the region D bounded by $z = x^2 + 3y^2$ and $z = 8 - x^2 - y^2$.

sol. We describe the region by type 1. First find the intersections of two surfaces. Set $x^2 + 3y^2 = 8 - x^2 - y^2$ to get $x^2 + 2y^2 = 4$. The the domain is the ellipse $x^2 + 2y^2 = 4$.

$$-2 \le x \le 2, \quad -\sqrt{(4-x^2)/2} \le y \le \sqrt{(4-x^2)/2}, \quad x^2+3y^2 \le z \le 8-x^2-y^2.$$

$$\begin{aligned} V(D) &= \iiint_D dzdxdy = \int_{-2}^{2} \left[2\int_0^{\sqrt{(4-x^2)/2}} (8-2x^2-4y^2)dy \right] dx \\ &= \int_{-2}^{2} \left[2(8-2x^2)y - \frac{4}{3}y^3 \right]_0^{\sqrt{(4-x^2)/2}} dx \\ &= 8\pi\sqrt{2}. \end{aligned}$$

□

Example 14.5.8. Find the common region of two cylinders (Figure **??**) $x^2 + y^2 \le 1$, $x^2 + z^2 \le 1$ $(z \ge 0)$.

sol.

$$\begin{aligned}\iint_{x^2+y^2\le 1}\int_0^{\sqrt{1-x^2}} dzdxdy &= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{1-x^2}dydx \\ &= 2\int_{-1}^{1}(1-x^2)dx \\ &= 2\left[x-\frac{x^3}{3}\right]_{-1}^{1} = 4(1-\frac{1}{3}) = \frac{8}{3}.\end{aligned}$$

□

14.6 Mass, Moments and Center of Mass

14.7 Triple integrals in Cylindrical and Spherical Coordinate

Cylindrical coordinate system

Given a point $P = (x, y, z)$, we can use polar coordinate for (x, y)-plane. Then it holds that

$$\text{Cylindrical to Cartesain} \begin{cases} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{cases}$$

We say (r, θ, z) is **cylindrical coordinate** of P.

Example 14.7.1. Identify the surface given by the equation $z = 2r$ in cylindrical coordinate.

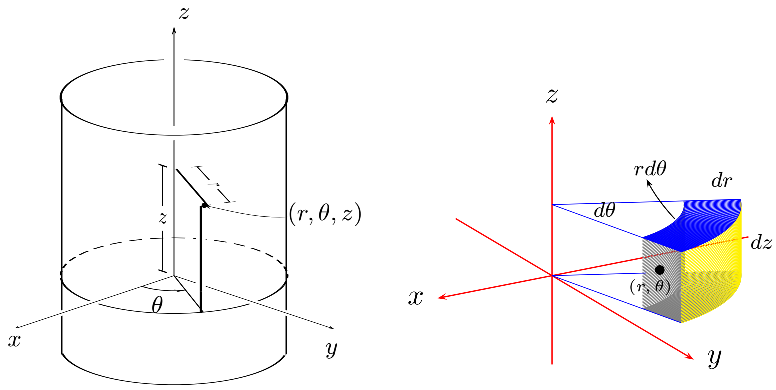
sol. Squaring, we have $z^2 = 4r^2 = 4(x^2 + y^2)$. The section $z = c$ is $c^2 = 4(x^2 + y^2)$, while with $x = 0$ we have $z = \pm y$. With $y = 0$ we have $z = \pm x$. Thus this is a cone.

□

Example 14.7.2. Change the equation $x^2 + y^2 - z^2 = 1$ to cylindrical coordinate.

sol. $r^2 - z^2 = 1$.

□



A sector of a cylinder

Figure 14.9: cylindrical coordinate

14.7.1 Integration in Cylindrical Coordinate

Let D be any region in $\mathbb{R}^3$. We describe it using the coordinate

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z.$$

We partition the region D into small cylindrical wedges (Fig **??**); Small wedge given by

$$[r_k, r_k + \Delta r_k] \times [\theta_k, \theta_k + \Delta\theta_k] \times [z_k, z_k + \Delta z_k]$$

has volume $\Delta V_k = \Delta A_k \Delta z_k \doteq r_k \Delta r_k \Delta\theta_k \Delta z_k$. So the sum $\sum_k f(x_k, y_k, z_k)\Delta V_k$ approaches

$$\iiint_D f(x, y, z)\, dxdydz = \iiint_{D^*} f(r\cos\theta, r\sin\theta, z) r\, dzdrd\theta. \qquad (14.2)$$

Here D^* is the region of described by the cylindrical coordinate (r, θ, z).

14.7.2 Integration in spherical coordinate system

We call (ρ, ϕ, θ) to be the **spherical coordinate** of $P(x, y, z)$ if

(1) ρ is the distance from P to the origin

(2) ϕ is the angle that makes with positive z axis

(3) θ is the angle from cylindrical coordinate.

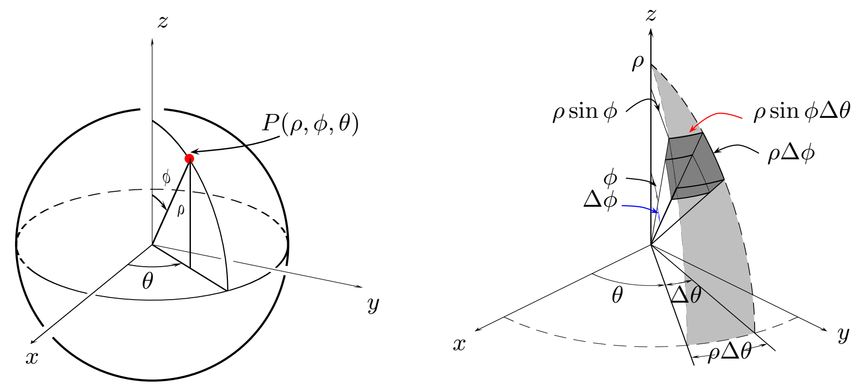


Figure 14.10: Spherical coordinate

For the point $P(x, y, z)$ we have

Spherical to Cartesian	$\begin{cases} x = \rho \sin \phi \cos \theta \\ y = \rho \sin \phi \sin \theta \\ z = \rho \cos \phi \end{cases}$	$\begin{pmatrix} \rho \geq 0 \\ 0 \leq \theta < 2\pi \\ 0 \leq \phi \leq \pi \end{pmatrix}$

Example 14.7.3. Express the surface (1) $xz = 1$ and (2) $x^2 + y^2 - z^2 = 1$ in spherical coordinate.

sol. (1) Since $xz = \rho^2 \sin \phi \cos \theta \cos \phi = 1$, we have the equation

$$\rho^2 \sin 2\phi \cos \phi = 2.$$

(2) Since $x^2+y^2-z^2 = x^2+y^2+z^2-2z^2 = \rho^2-2(\rho \cos \phi)^2 = \rho^2(1-2\cos^2 \phi)$, the equation is $\rho^2(1 - 2\cos^2 \phi) = 1$.

□

Volumes in Spherical Coordinate-Geometric Derivation

Consider the small region bounded by the following conditions: (Fig.**??**)

$$\rho_0 \leq \rho \leq \rho_0 + \Delta\rho, \quad \phi_0 \leq \phi \leq \phi_0 + \Delta\phi, \quad \theta_0 \leq \theta \leq \theta_0 + \Delta\theta.$$

The integral of f is defined as

$$\iiint_D f\,dV = \int\int\int f(\rho,\phi,\theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta. \tag{14.3}$$

How to integrate in Spherical coordinates

Let D be the region determined by

$$D = \{(\rho,\phi,\theta) : g_1(\phi,\theta) \le \rho \le g_2(\phi,\theta), h_1 \le \phi \le h_2, \alpha \le \theta \le \beta\}.$$

To evaluate $\iiint_D f\,dV = \int\int\int f(\rho,\phi,\theta)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$ we proceed as follows:

(1) Sketch the region D and project it onto xy plane.

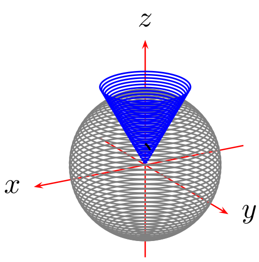
(2) Find the ρ limit of the integration $(g_1(\phi,\theta) \le \rho \le g_2(\phi,\theta))$

(3) Find the ϕ limit of the integration $(h_1(\theta) \le \phi \le h_2(\theta))$

(4) Find the θ limit of the integration

Example 14.7.4. Find the volume of the "ice cream cone" D cut from the solid $\rho \le 1$ by the cone $\phi = \pi/3$.

sol.



$\rho \le 1,\ \phi \le \pi/3$

$$\begin{aligned}
V &= \iiint_D \rho^2 \sin\phi d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3}\int_0^1 \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3} \left[\frac{\rho^3}{3}\right]_0^1 \sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/3} \frac{1}{3}\sin\phi\, d\phi\, d\theta \\
&= \int_0^{2\pi} \left[-\frac{1}{3}\cos\phi\right]_0^{\pi/3} d\theta \\
&= 2\pi\left(-\frac{1}{6}+\frac{1}{3}\right) = \frac{\pi}{3}.
\end{aligned}$$

□

Example 14.7.5. Compute

$$\iiint_W \exp(x^2+y^2+z^2)^{3/2} dV,$$

where W is the unit ball.

sol. By spherical coordinate,

$$\iiint_W \exp(x^2+y^2+z^2)^{3/2} dV = \iiint_{W^*} \rho^2 e^{\rho^3} \sin\phi d\theta\, d\phi\, d\rho.$$

Changing it to an iterated integral, we have

$$\begin{aligned} & \int_0^1 \int_0^\pi \int_0^{2\pi} \rho^2 e^{\rho^3} \sin\phi d\theta\, d\phi\, d\rho \\ = \ & 2\pi \int_0^1 \int_0^\pi \rho^2 e^{\rho^3} \sin\phi d\phi\, d\rho \\ = \ & 4\pi \int_0^1 \rho^2 e^{\rho^3} d\rho = \frac{4}{3}\pi(e-1). \end{aligned}$$

□

14.8 Substitution-Change of variables

Let $F(u,v) = f(x(u,v), y(u,v))$ and recalling the definition of integral, we see

$$\lim_{n\to\infty} \sum_{i=1}^n f(x_i, y_i)\Delta A_i(x,y) = \lim_{n\to\infty} \sum_{i=1}^n F(u_i, v_i)\Delta A_i(u,v). \tag{14.4}$$

One-to-one map and onto map

Example 14.8.1. Let D be the region in the first quadrant lying between concentric circles $r = a, r = b$ and $\theta_1 \le \theta \le \theta_2$. (Fig. **??**) Let

$$T(r,\theta) = (r\cos\theta, r\sin\theta)$$

be the polar coordinate map. Find a region D^* in (r,θ) coordinate plane such that $D = T(D^*)$.

sol. In D, we see

$$a \le r \le b, \quad \theta_1 \le \theta \le \theta_2.$$

Hence

$$D^* = [a,b] \times [\theta_1, \theta_2].$$

□

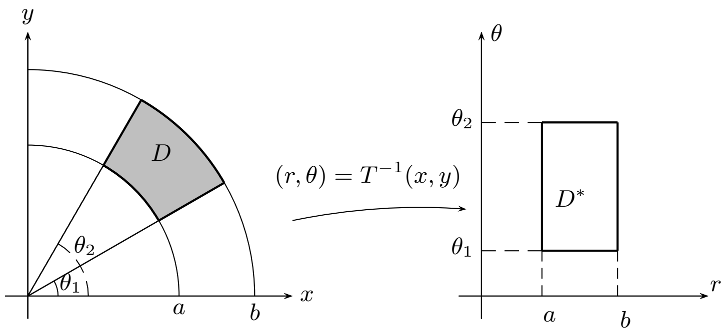


Figure 14.11: Inverse image of a polar rectangle

Coordinate transformations

Let D^* be a region in $\mathbb{R}^2$. Suppose T is C^1-map $D^* \to \mathbb{R}^2$. We denote the image by $D = T(D^*)$. (Fig **??**)

$$T(D^*) = \{(x, y) \mid (x, y) = T(u, v), \quad (u, v) \in D^*\}.$$

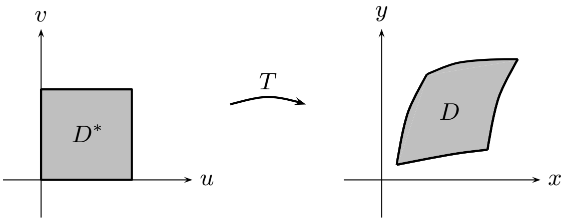


Figure 14.12: The transformation T maps D^* to D

Jacobian Determinant-measures change of area

We first see how the area of a region changes under a linear map. Let $D^* = [0, 1] \times [0, 1]$, and construct a linear map T that maps D^* onto a parallelogram D. Consider the vector $\mathbf{c}_1 := \mathbf{a}_2 - \mathbf{a}_1$, $\mathbf{c}_2 := \mathbf{a}_4 - \mathbf{a}_1$, and set (one may assume $\mathbf{a}_1 = 0$)

$$T(u, v) = \mathbf{c}_1 u + \mathbf{c}_2 v.$$

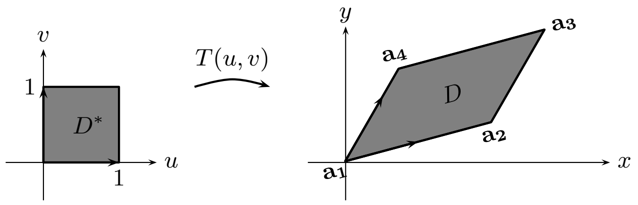


Figure 14.13: The image of a rectangle under a linear transform T

The two tangent vectors to D at the origin are

$$\begin{aligned} T_u &= \mathbf{a}_2 \\ T_v &= \mathbf{a}_4. \end{aligned}$$

The area of the parallelogram D is

$$Area(D) = \|(\mathbf{a}_2 - \mathbf{a}_1) \times (\mathbf{a}_4 - \mathbf{a}_1)\| = |J|,$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} := det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = |DT|.$$

J is called the **Jacobian of** T.

Thus for the area change, we have

Theorem 14.8.2. *Let A be a 2×2 matrix with non zero determinant. Let T be a linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$. Then T maps a parallelogram D^* onto the parallelogram $D = T(D^*)$ and*

$$Area\ of\ D = |det\ A| \cdot (Area\ of\ D^*).$$

Example 14.8.3. Let T be $((x+y)/2, (x-y)/2)$ and let D be the square whose vertices are $(1,0), (0,1), (-1,0), (0,-1)$. Find a D^* such that $D = T(D^*)$.

sol. Since T is linear $T(\mathbf{x}) = A\mathbf{x}$ where A is 2×2 matrix whose determinant is nonzero. T^{-1} is also a linear transform. Hence by Theorem **??**, D^* must be a parallelogram. To find D^*, it suffices to find the inverse image of vertices. It turns out that

$$D^* = [-1,1] \times [-1,1].$$

Now

$$A(D) = (\sqrt{2})^2 = 2, \ |det\, A| = \frac{1}{2}, \ A(D^*) = 4, .$$

□

Change of variable in the definite integrals

Let $D = T(D^*)$, where

$$T(u, v) = (x(u, v), y(u, v)) \text{ for } (u, v) \in D^*.$$

Then we have

$$\iint_D f(x, y)\, dxdy \quad = \quad \iint_{D^*} f(T(u, v)) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| dudv. \tag{14.5}$$

Example 14.8.4. Evaluate

$$\int_0^1 \int_0^{1-x} \sqrt{x+y}(y - 2x)^2 dydx.$$

sol. Let us use the substitution $u = x + y, v = y - 2x$, so that

$$x = \frac{u}{3} - \frac{v}{3}, \quad y = \frac{2u}{3} + \frac{v}{3}. \tag{14.6}$$

One can find the limits of integration and find $J(u, v) = \frac{1}{3}$. To find the limit of integration, we see Figure **??**. and Table **??**.

Table 14.1: Limit of integration for Example **??**

xy eq. for boundary	uv eq. for boundary	Simplified
$x + y = 1$	$\frac{u-v}{3} + \frac{2u+v}{3} = 0$	$u = 1$
$x = 0$	$\frac{u}{3} - \frac{v}{3} = 0$	$v = u$
$y = 0$	$\frac{2u+v}{3} = 0$	$v = -2u$

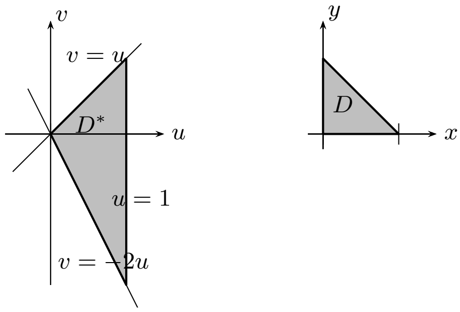


Figure 14.14: Change of variables for Example **??**

Hence we obtain

$$\begin{aligned}\int_0^1 \int_0^{1-x} \sqrt{x+y}(y-2x)^2 dydx &= \int_0^1 \int_{v=-2u}^{v=u} \sqrt{u} v^2 |J(u,v)| dvdu \\ &= \frac{1}{3}\int_0^1 \sqrt{u}\left[\frac{v^3}{3}\right]_{-2u}^{u} du \\ &= \frac{1}{9}\int_0^1 \sqrt{u}(u^3+8u^3) du \\ &= \int_0^1 u^{7/2} du = \frac{2}{9}.\end{aligned}$$

□

Example 14.8.5. Evaluate

$$\int_1^2 \int_{1/y}^{y} \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dxdy.$$

sol. We use the substitution $u = \sqrt{xy}, v = \sqrt{\frac{y}{x}}$, so that

$$x = \frac{u}{v}, \quad y = uv, u, v > 0. \tag{14.7}$$

We see

$$J(u,v) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{vmatrix} = \frac{2u}{v}.$$

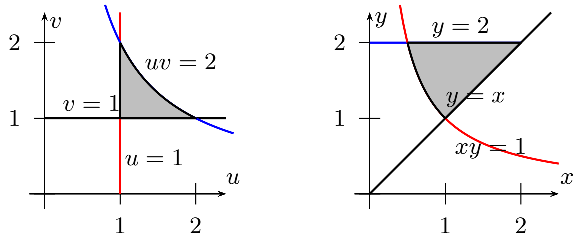


Figure 14.15: Change of variables for Example ??

Table 14.2: Limit of integration for Example ??

xy eq. for boundary	uv eq. for boundary	Simplified
$y = x$	$uv = \frac{u}{v}$	$v = 1(u > 0)$
$xy = 1$	$u = 1$	$u = 1$
$y = 2$	$u = \sqrt{2x},\ v = \sqrt{\frac{2}{x}}$	$uv = 2$

(Note that if we integrate w.r.t u first, we run into trouble!) Once we find the limits of integration(need the region D and D^*) from Table ??, we obtain

$$\begin{aligned}
\iint_R \sqrt{\frac{y}{x}} e^{\sqrt{xy}} dxdy &= \iint_R ve^u \frac{2u}{v} dudv \\
&= \int_1^2 \int_1^{2/u} 2ue^u dvdu \\
&= 2\int_1^2 [vue^u]_{v=1}^{v=2/u} du \\
&= 2\int_0^1 (2e^u - ue^u) du \\
&= 2\left[(2e^u - ue^u) + e^u\right]_{u=1}^{u=2} = 2e(e-2).
\end{aligned}$$

□

Change of variable formula - general case

Let T be a differentiable mapping from a subset of $\mathbb{R}^2$ to $\mathbb{R}^2$. Let $D^* = [u_0, u_0 + \Delta u] \times [v_0, v_0 + \Delta v]$ and D be the image of D^* under T. Consider

$$T(u,v) = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(u_0,v_0) + \frac{\partial x}{\partial u}(u_0,v_0)\Delta u + \frac{\partial x}{\partial v}(u_0,v_0)\Delta v + h.o.t \\ y(u_0,v_0) + \frac{\partial y}{\partial u}(u_0,v_0)\Delta u + \frac{\partial y}{\partial v}(u_0,v_0)\Delta v + h.o.t \end{bmatrix} \tag{14.8}$$

or in vector form, we have

$$T \begin{bmatrix} u \\ v \end{bmatrix} = \mathbf{X} = \mathbf{X}_0 + DT \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} + h.o.t$$

and replace the map T by its linear part DT.

Geometric meaning of DT

Let

$$T_u := DT(u,v) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{bmatrix}$$

and

$$T_v := DT(u,v) \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{bmatrix}.$$

Now the two tangent vectors $T_u \Delta u, \quad T_v \Delta v$ form a parallelogram approximating the region D(Figure **??**). Hence the area of the parallelogram is

$$\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v = \frac{\partial(x,y)}{\partial(u,v)} \Delta u \Delta v \doteq J \cdot A(D^*).$$

$$\|T_u \times T_v\| \Delta u \Delta v = |J| \Delta u \Delta v.$$

Summing over all subregions and taking the limit as $\Delta u, \Delta v \to 0$ we obtain the formula.

Change of Variables in Triple Integrals

Definition 14.8.6. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$T(u,v,w) = (x(u,v,w), y(u,v,w), z(u,v,w)).$$

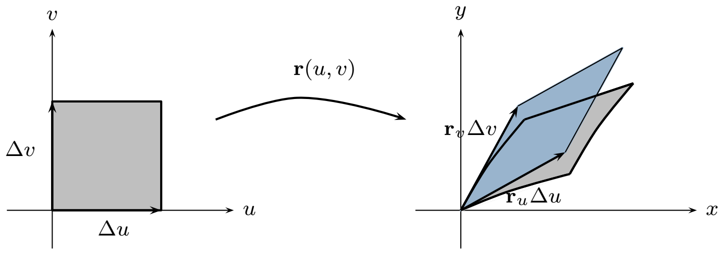


Figure 14.16: approximate $T(D^*)$

The the **Jacobian** J is again, as 2D case, the determinant of the derivative DT

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v}, & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v}, & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u}, & \frac{\partial z}{\partial v}, & \frac{\partial z}{\partial w} \end{bmatrix}.$$

The absolute value of this determinant is equal to the volume of parallelepiped determ'd by the following vectors

$$\begin{aligned} \mathbf{T}_u &= \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \\ \mathbf{T}_v &= \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k} \\ \mathbf{T}_w &= \frac{\partial x}{\partial w}\mathbf{i} + \frac{\partial y}{\partial w}\mathbf{j} + \frac{\partial z}{\partial w}\mathbf{k}, \end{aligned}$$

which is the absolute value of the triple product (recall Chap. 12.4)

$$|(\mathbf{T}_u \times \mathbf{T}_v) \cdot \mathbf{T}_w| = |J|.$$

Caution: Three vectors $\mathbf{T}_u, \mathbf{T}_v, \mathbf{T}_w$ are column vectors of DT, but since $det(A) = det(A^T)$ for any square matrix, we have

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \mathbf{T}_u, & \mathbf{T}_v, & \mathbf{T}_w \end{bmatrix}.$$

Theorem 14.8.7. *If T is a $\mathcal{C}^1$- map from D^* onto D in $\mathbb{R}^3$ and $f : D \subset$*

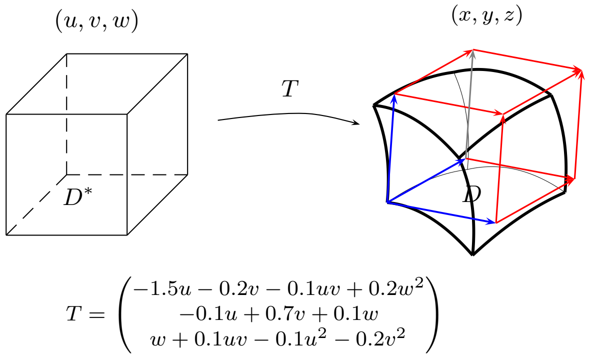


Figure 14.17: Deformed box and parallelepiped generated by tangent vectors.

$\mathbb{R}^3 \to \mathbb{R}$ *is continuous, then*

$$\iiint_D dxdydz = \iiint_{D^*} |J|\, dudvdw, \tag{14.9}$$

$$\iiint_D f(x,y,z)\, dxdydz = \iiint_{D^*} f(T(u,v,w))|J|\, dudvdw. \tag{14.10}$$

Example 14.8.8. Evaluate

$$\int_0^3 \int_0^4 \int_{y/2}^{y/2+1} \left(\frac{2x-y}{2} + \frac{z}{3}\right) dxdydz$$

using the transformation

$$u = (2x-y)/2, \quad v = y/2, \quad w = z/3. \tag{14.11}$$

sol. We see

$$x = u + v, \quad y = 2v, \quad z = 3w. \tag{14.12}$$

We see

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u}, & \frac{\partial x}{\partial v}, & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u}, & \frac{\partial y}{\partial v}, & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u}, & \frac{\partial z}{\partial v}, & \frac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6.$$

One can find the limits of integration we obtain

Table 14.3: Limit of integration for Example **??**

xyz eq. for boundary	uvw eq. for boundary	Simplified eq.
$x = y/2$	$u + v = 2v/2$	$u = 0$
$x = y/2 + 1$	$u + v = 2v/2 + 1$	$u = 1$
$y = 0$	$2v = 0$	$v = 0$
$y = 4$	$2v = 4$	$v = 2$
$z = 0$	$3w = 0$	$w = 0$
$z = 3$	$3w = 3$	$w = 1$

$$\begin{aligned}
\iiint_D f\,dxdydz &= \int_0^1 \int_0^2 \int_0^1 (u + w)\,|J| dudvdw \\
&= 6 \int_0^1 \int_0^2 \left[\frac{u^2}{2} + uw \right]_0^1 dvdw \\
&= 6 \int_0^1 \int_0^2 \left(\frac{1}{2} + w \right) dvdw \\
&= 6 \int_0^1 (1 + 2w)\,dw = 12.
\end{aligned}$$

□

Spherical Coordinate - revisited

Example 14.8.9. Derive the integration formula in spherical coordinate using Theorem **??**.

sol. Spherical coordinate is given by

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

The Jacobian of the mapping $(\rho, \phi, \theta) \to (x, y, z)$ is

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} &= \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} \\
&= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= \rho^2\sin\phi(\cos^2\phi + \sin^2\phi) = \rho^2\sin\phi.
\end{aligned}$$

Hence

$$\iiint_D f(x,y,z)\,dxdydz = \iiint_{D^*} F(\rho,\phi,\theta)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Here $F(\rho,\phi,\theta)$ means $f(x(\rho,\phi,\theta), y(\rho,\phi,\theta), z(\rho,\phi,\theta))$.

□

Chapter 15

Integral of Vector Fields

15.1 Line Integrals

Line integral(Path integral) of a scalar function

Let C be a C^1- curve $\mathbf{x}(t) = \mathbf{r}(t) = (x(t), y(t), z(t)) : [a, b] \to C \subset \mathbb{R}^3$. Let $P\colon a = t_0 < t_1 < \cdots < t_k = b$ be the partition of $[a, b]$. Then the Riemann sum of $f : C \to \mathbb{R}$ is

$$\sum_{i=1}^{k} f(\mathbf{x}(t_i^*))\Delta s_i = \sum_{i=1}^{k} f(\mathbf{x}(t_i^*))\|\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})\| = \sum_{i=1}^{k} f(\mathbf{x}(t_i^*))\Delta s_i.$$

Definition 15.1.1. We define **the line integral** of f over C as:

$$\int_C f(x, y, z)ds = \int_a^b f(g(t), h(t), k(t))\|\mathbf{v}(t)\|dt = \int_a^b f(\mathbf{x}(t))\|\mathbf{x}'(t)\|\,dt.$$

Here $s(t)$ is the arc length parameter:

$$s(t) = \int_0^t \|\mathbf{v}(\tau)\|d\tau$$

Example 15.1.2. Find path integral of $f(x, y, z) = x^2 + y^2 + z^2$ over C where

$$\mathbf{x}(t) = (\cos t, \sin t, t), \quad t \in [0, 2\pi].$$

sol. Since $\mathbf{x}'(t) = (-\sin t, \cos t, 1)$, the line integral is

$$\begin{aligned}\int_C f\,ds &= \int_0^{2\pi} f(\mathbf{x}(t))\|\mathbf{x}'(t)\|\,dt \\ &= \int_0^{2\pi} (\cos^2 t + \sin^2 t + t^2)\|(-\sin t, \cos t, 1)\|\,dt \\ &= \int_0^{2\pi} (1+t^2)\sqrt{2}\,dt \\ &= \sqrt{2}\left(2\pi + 8\pi^3/3\right).\end{aligned}$$

□

Mass and Moment of a wire

Imagine coils or springs and wires as masses distributed along smooth curves in space.

When a curve C is parameterized by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, a \le t \le b$, the density of wire is $\delta(x(t), y(t), z(t))$.

$M = \int_C \delta\,ds$
$M_{yz} = \int_C x\delta\,ds$
$M_{zx} = \int_C y\delta\,ds$
$M_{xy} = \int_C z\delta\,ds$
$\bar{x} = \frac{M_{yz}}{M}, \quad \bar{y} = \frac{M_{zx}}{M}, \bar{z} = \frac{M_{xy}}{M}.$
moment of inertia about the axis and the line L
$I_x = \int_C (y^2+z^2)\delta\,ds,\ I_y = \int_C (x^2+z^2)\delta\,ds,\ I_z = \int_C (x^2+y^2)\delta\,ds,\ I_L = \int_C r^2\delta\,ds.$

15.2 Line integral of Vector fields: Work, Circulation and Flux

Vector fields, Gradient fields and potentials

Given real C^1- function $f(x_1, x_2, \ldots, x_n)$, we define the **gradient field** by

$$\nabla f := (\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}).$$

f is called the **potential function**.

Line Integrals of Vector Fields

The Rieman sum of a vector field along a curve is a work defined by

$$\sum_{i=0}^{n-1} \mathbf{F}(\mathbf{x}(t_i^*)) \cdot \Delta \mathbf{x}_i = \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{x}(t_i)) \cdot [\mathbf{x}(t_i + \Delta t) - \mathbf{x}(t_i)].$$

Taking the limit

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{x}(t_i^*)) \cdot \Delta \mathbf{x}_i &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \mathbf{F}(\mathbf{x}(t_i)) \cdot \frac{\Delta \mathbf{x}_i}{\Delta t} \Delta t \\ &= \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt. \end{aligned}$$

$$\begin{aligned} \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{x}'(t) dt &= \int_a^b \left[\mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathbf{x}'(t)}{\|\mathbf{x}'(t)\|} \right] \|\mathbf{x}'(t)\| dt \\ &= \int_a^b [\mathbf{F}(\mathbf{x}(t)) \cdot \mathbf{T}(t)] \, \|\mathbf{x}'(t)\| dt \\ &= \int_C (\mathbf{F} \cdot \mathbf{T}) \, ds \equiv \int_C \mathbf{F} \cdot \, d\mathbf{x}. \end{aligned}$$

Line integral with resp. to dx, dy or dz

Suppose the vector field

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

is given and

$$\mathbf{r}(t) \equiv \mathbf{x}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}, \;\; a \le t \le b$$

is a smooth curve.Then recalling $\mathbf{r}'(t) = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k}$, we see

$$\int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = \int_a^b (M, N, P) \cdot (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}) dt = \int_C M dx + N dy + P dz. \quad (15.1)$$

Flow integrals and circulation of velocity fields

Definition 15.2.1. If $\mathbf{F}$ is a continuous vector field and $\mathbf{T}$ is unit tangent vector on C, then the **flow** of $\mathbf{F}$ along C is

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

If the curve is closed, then the flow is called the **circulation** of $\mathbf{F}$ along C.

Example 15.2.2. Let $\mathbf{F}(x,y,z) = x\mathbf{i} + z\mathbf{j} + y\mathbf{k}$. Find the flow of $\mathbf{F}$ along the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k},\ 0 \le t \le \pi/2$.

sol.

$$\frac{d\mathbf{r}}{dt} = (-\sin t, \cos t, 1).$$

$$\begin{aligned}\int_C \mathbf{F}\cdot\frac{d\mathbf{r}}{dt} &= \int_0^{\pi/2} (-\sin t\cos t + t\cos t + \sin t)\,dt \\ &= \left[\frac{\cos^2 t}{2} + t\sin t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{2} - \frac{1}{2}.\end{aligned}$$

□

Flux across a simple closed plane curve

Definition 15.2.3. If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F}$ and $\mathbf{n}$ is unit outward normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds.$$

Calculating flux across a simple closed plane curve:

Let $(x(t), y(t))$ be a parametrization of C and $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$. Then the unit tangent vector is $\mathbf{T} = \frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j}$, and unit normal vector is

$$\mathbf{n} = \frac{dy}{ds}\mathbf{i} - \frac{dx}{ds}\mathbf{j}.$$

Hence the flux is

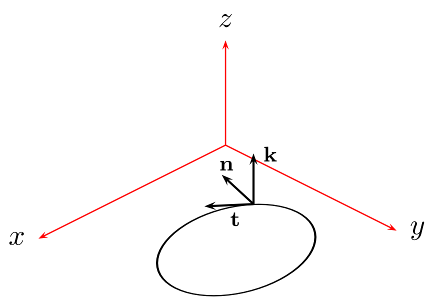


Figure 15.1: Outward normal $\mathbf{n} = \mathbf{t} \times \mathbf{k}$ directs the rhs of a walking man

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \int_C \left(M\frac{dy}{ds} - N\frac{dx}{ds}\right) ds = \oint_C M dy - N dx. \qquad (15.2)$$

Example 15.2.4. Find the flux of $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + x\mathbf{j}$ along the circle $x^2 + y^2 = 1$. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$ $(0 \le t \le 2\pi)$.

sol. We see $\frac{d\mathbf{r}}{dt} = (-\sin t, \cos t)$. Hence

$$dy = \cos t, \quad dx = \sin t.$$

Since

$$M = x - y = \cos t - \sin t,\ \ N = x = \cos t$$

we see the flux is

$$\begin{aligned}\int_C M dy - N dx &= \int_0^{2\pi} (\cos^2 t - \sin t\cos t + \sin t\cos t)\,dt \\ &= \int_0^{2\pi} \cos^2 t\,dt = \int_0^{2\pi} \frac{1+\cos 2t}{2} dt \\ &= \left[\frac{t}{2} + \frac{\sin 2t}{4}\right]_0^{2\pi} = \pi.\end{aligned}$$

□

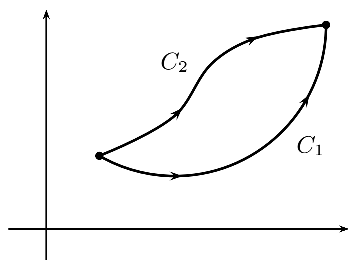


Figure 15.2: Two curves having the same end points

15.3 Path independence, conservative vector fields

Definition 15.3.1. A line integral a vector field $\mathbf{F}$ is called **path independent** if

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r} \tag{15.3}$$

for any two oriented curves C_1, C_2 lying in the domain of $\mathbf{F}$ having same end points. The field is called **conservative**.

A vector field $\mathbf{F}$ is called a **gradient vector field** if $\mathbf{F} = \nabla f$ for some real valued function f. Thus

$$\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

The function f is called a **potential** of $\mathbf{F}$.

Example 15.3.2. A gravitational force field has the potential function $f = \frac{GmM}{r}$ $(\mathbf{r} = (x, y, z),\ r = \sqrt{x^2 + y^2 + z^2})$.

$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r} = \nabla f.$$

sol. We take derivative of $r^2 = x^2 + y^2 + z^2$, i.e., $2r\frac{\partial r}{\partial x} = 2x, 2r\frac{\partial r}{\partial y} = 2y, 2r\frac{\partial r}{\partial z} = 2z$. Thus

$$\nabla f = -\frac{GmM}{r^2}(\frac{\partial r}{\partial x}, \frac{\partial r}{\partial y}, \frac{\partial r}{\partial z}) = -\frac{GmM}{r^3}\mathbf{r}.$$

□

Theorem 15.3.3. *Suppose* $f : \mathbb{R}^3 \to \mathbb{R}$ *is class* C^1 *and* $\mathbf{r} : [a,b] \to \mathbb{R}^3$ *is smooth curve* C *and* $\mathbf{F}$ *is a continuous gradient field such that* $\mathbf{F} = \nabla f$. *Then*

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_a^b \frac{d}{dt} f(\mathbf{r}(t))dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

In other words, the gradient field is conservative.

Definition 15.3.4. A region R in $\mathbb{R}^2$ or $\mathbb{R}^3$ is called **simply connected** if every closed curve C in R can be continuously shrunk to a point (contractible) while remaining in R throughout the deformation.

Curl of a vector field in $\mathbb{R}^3$

If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = (F_1, F_2, F_3)$, then $\nabla \times \mathbf{F}$ ($\equiv \mathbf{curl}\,\mathbf{F}$) is defined as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix} = \left(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\right)\mathbf{i} + \left(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\right)\mathbf{j} + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}$$

Theorem 15.3.5. ***(Conservative Field)*** *Let* $\mathbf{F}$ *be a* $\mathcal{C}^1$*-vector field on a simply connected domain in* $\mathbb{R}^3$. *Then the following conditions are equivalent:*

(1) For any oriented closed curve C, $\int_C \mathbf{F} \cdot d\mathbf{x} = 0$.

(2) For any two oriented curve C_1, C_2 *having same end points,*

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{x} = \int_{C_2} \mathbf{F} \cdot d\mathbf{x}.$$

(3) $\mathbf{F}$ *is the gradient of some function* f, *i.e,* $\mathbf{F} = \nabla f$.

(4) $\nabla \times \mathbf{F} = \mathbf{0}$.

Component test for conservative field

If a field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative on a simply connected domain, then by above Theorem, there exists some function f s.t.

$$\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Hence we can check the following holds: (by taking the derivative)

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \text{ and } \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{15.4}$$

Example 15.3.6. Show that the vector field is conservative and find its potential.

$$\mathbf{F}(x, y, z) = (e^x \sin y - yz)\mathbf{i} + (e^x \cos y - xz)\mathbf{j} + (z - xy)\mathbf{k}.$$

sol. One can check (**??**) or check if the curl $\mathbf{F}$ is zero:

$$\begin{aligned}
\nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x \sin y - yz & e^x \cos y - xz & z - xy \end{vmatrix} \\
&= \left(\frac{\partial}{\partial y}(z - xy) - \frac{\partial}{\partial z}(e^x \cos y - xz) \right) \mathbf{i} + \left(\frac{\partial}{\partial z}(e^x \sin y - yz) - \frac{\partial}{\partial x}(z - xy) \right) \mathbf{j} \\
&+ \left(\frac{\partial}{\partial x}(e^x \cos y - xz) - \frac{\partial}{\partial y}(e^x \sin y - yz) \right) \mathbf{k} = \mathbf{0}.
\end{aligned}$$

So the condition (**??**) holds. To find a potential we need to find and f satisfying

$$\frac{\partial f}{\partial x} = e^x \sin y - yz, \quad \frac{\partial f}{\partial y} = e^x \cos y - xz, \quad \frac{\partial f}{\partial z} = z - xy. \tag{15.5}$$

Thus we proceed as follows: First integrate w.r.t x.

(1) $f(x, y, z) = \int (e^x \sin y - yz) dx = e^x \sin y - xyz + g(y, z)$ for some $g(y, z)$.

(2) $\frac{\partial f}{\partial y} = e^x \cos y - xz + \frac{\partial g}{\partial y} = e^x \cos y - xz$. Thus $g(y, z)$ is a function of z only, thus $g = g(z)$. Taking derivative of f w.r.t z, we have

(3) $\frac{\partial f}{\partial z} = -xy + g'(z) = z - xy$. Thus $g(z) = \frac{1}{2}z^2 + C$.

(4) Hence $f(x, y, z) = e^x \sin y - xyz + \frac{1}{2}z^2 + C$.

□

Exact differential form

The expression $F_1dx + F_2dy + F_3dz$ is called a **differential form**. We can compute the line integral of a differential form as

$$\int_C F_1dx + F_2dy + F_3dz \quad = \quad \int_a^b \big(F_1x'(t) + F_2y'(t) + F_3z'(t)\big)\,dt.$$

Definition 15.3.7. A differential form is said to be **exact** if it has the form

$$Mdx + Ndy + Pdz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \equiv df = \nabla f \cdot d\mathbf{x}.$$

for some scalar function f.

Component test for exactness

The differential form is exact if and only if (following Theorem **??**)

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \text{ and } \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{15.6}$$

This is a consequence of Theorem **??** for conservative field.

Example 15.3.8. Find the potential of the vector field if it is conservative.

$$\mathbf{F}(x, y) = (2xy + \cos 2y)\mathbf{i} + (x^2 - 2x\sin 2y)\mathbf{j}.$$

sol.

First we check that $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$. Hence it is conservative. Let f be the potential function. Then it satisfies $\nabla f = \mathbf{F}$, i.e.,

$$\frac{\partial f}{\partial x} = 2xy + \cos 2y, \quad \frac{\partial f}{\partial y} = x^2 - 2x\sin 2y. \tag{15.7}$$

Thus we proceed as follows:

(1) Integrate: $f(x, y) = \int \frac{\partial f}{\partial x}\,dx = \int 2xy + \cos 2y\,dx = x^2y + x\cos 2y + g(y)$

(2) Set $\frac{\partial f}{\partial y} = x^2 - 2x\sin 2y + g'(y)$

(3) Show $g(x, y) = C$.

Thus we see $f(x, y) = x^2 - 2x\sin 2y + C$.

□

Example 15.3.9. Show the form $ydx + xdy + 4dz$ is exact and evaluate the integral

$$\int_C ydx + xdy + 4dz.$$

sol. ...

□

15.4 Green's Theorem in the plane

Circulation and flux

(1) The **circulation rate** measures the spin of the fluid around a closed curve, which is given $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C Mdx + Ndy$.

(2) The **flux rate** measures the rate at which the fluid leaves out of the closed curve, which is given $\oint_C \mathbf{F} \cdot \mathbf{n}ds = \oint_C Mdy - Ndx$.

$$\oint_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

$$\oint_C Mdy - Ndx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dxdy.$$

Relation with 3D curl

If $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ is two dimensional vector field, then it can be considered as a three dimensional vector field as $\mathbf{F} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j} + 0 \cdot \mathbf{k}$. The **curl** $\mathbf{F}$ can be computed :

$$\begin{aligned} \mathbf{curl}\ \mathbf{F} &= \Big(\frac{\partial P}{\partial y} - \frac{\partial N}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial M}{\partial z} - \frac{\partial P}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k} \\ &= \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big)\mathbf{k}. \end{aligned}$$

Definition 15.4.1. The **circulation density** of $\mathbf{F}$ is the expression $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$, also called the $\mathbf{k}$ - component of the curl denoted by $(\mathrm{curl}\, \mathbf{F}) \cdot \mathbf{k}$.

Physical meaning:

(1) The integral of a circulation around a closed curve is the same as the integral of the curl of **F** on the region enclosed by the curve.

(2) Normal component of **curl F** is the rate of **rotation** along the plane.

Green's Theorem

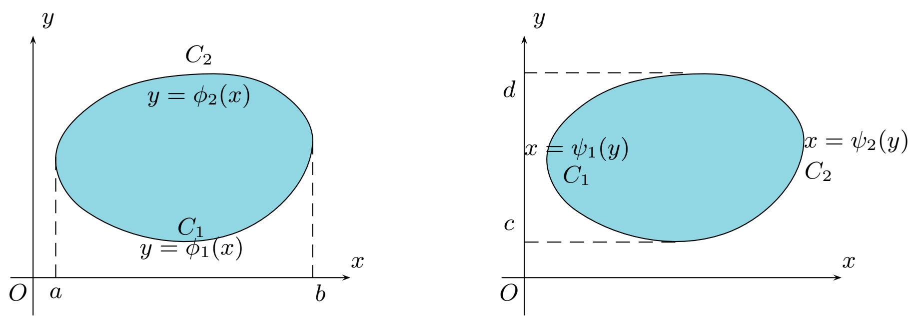


Figure 15.3: As type 1 region and boundary

Theorem 15.4.2. ***(Green's theorem: Circulation-Curl form)*** *Let D be a closed bounded, region in $\mathbb{R}^2$ with boundary ∂D Then*

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{T}\, ds = \oint_{\partial D} M\, dx + N\, dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

The integral of the **circulation** *around a ∂D is the integral of* **curl F·k** *on D.*

Proof. Assume D is a region of type 1 given as follows:

$$D = \{(x, y)|\ a \le x \le b, \phi_1(x) \le y \le \phi_2(x)\}.$$

We decompose the boundary of D as $\partial D = C_1^+ + C_2^-$(fig **??**). Using the

Fubini's theorem, we can evaluate the double integral as an iterated integral

$$\begin{aligned}\iint_D -\frac{\partial M(x,y)}{\partial y} dxdy &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} -\frac{\partial M(x,y)}{\partial y} dydx \\ &= \int_a^b [M(x,\phi_1(x)) - M(x,\phi_2(x))]dx.\end{aligned}$$

On the other hand, C_1^+ can be parameterized as $x \to (x, \phi_1(x)), a \le x \le b$ and C_2^+ can be parameterized as $x \to (x, \phi_2(x)), a \le x \le b$. Hence

$$\int_a^b M(x,\phi_i(x))dx = \int_{C_i^+} M(x,y)dx, \quad i = 1, 2.$$

By reversing orientations

$$-\int_a^b M(x,\phi_2(x))dx = \int_{C_2^-} M(x,y)dx.$$

Hence

$$\iint_D -\frac{\partial M}{\partial y}\, dxdy = \int_{C_1^+} M\, dx + \int_{C_2^-} M\, dx = \int_{\partial D} M\, dx.$$

Similarly if D is a region of type 2, one can show that

$$\iint_D \frac{\partial N}{\partial x} dxdy = \int_{C_1^+} N dy + \int_{C_2^-} N dy = \int_{\partial D} N\, dy.$$

Here C_1 and C_2 are the curves defined by $x = \psi_1(y)$ and $x = \psi_2(y)$ for $c \le y \le d$. The proof is completed. □

Theorem 15.4.3. ***(Green's theorem: Flux-Divergence form)*** *Let D be a closed bounded, region in $\mathbb{R}^2$ with boundary $C = \partial D$ with positive orientation.*

Suppose $\mathbf{F}(x,y) = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}$ be a vector field of class $\mathcal{C}^1$. Then

$$\oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, ds = \oint_{\partial D} M\, dy - N\, dx = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy.$$

The integral of the outward **flux** *around a ∂D = the integral of div*$\mathbf{F}$ *on D.*

Example 15.4.4. Verify Green's theorem for

$$M(x,y) = \frac{-y}{x^2+y^2}, \quad N(x,y) = \frac{x}{x^2+y^2}$$

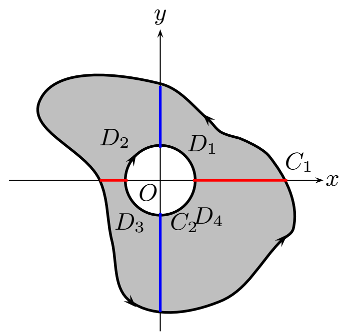


Figure 15.4: Apply Green's theorem to each of the regions

on $D = \{(x, y) |\ h^2 \le x^2 + y^2 \le 1\}$, $0 < h < 1$.

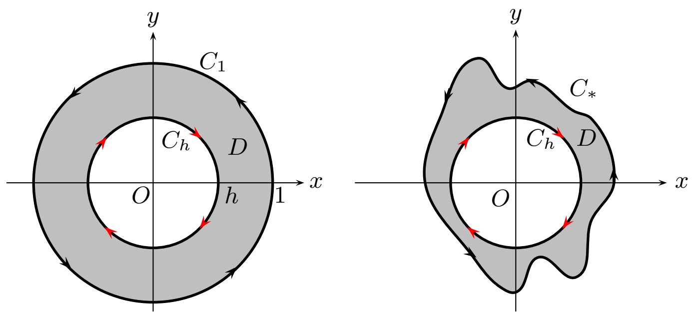


Figure 15.5: Domains for Example **??** and Example **??**

sol. The boundary of D consists of two circles.

$$C_1 : x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi$$
$$C_h : x = h \cos t, \qquad y = h \sin t, \qquad 0 \le t \le 2\pi.$$

In the curve $\partial D = C_h \cup C_1$, C_1 is oriented counterclockwise while C_h is oriented clockwise. Since M, N are class $\mathcal{C}^1$ in the annuls D, we can use Green's theorem. Since

$$\frac{\partial M}{\partial y} = \frac{(x^2+y^2)(-1)+2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}$$

we have

$$\iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \int_D 0\ dxdy = 0.$$

On the other hand,

$$\begin{aligned}\int_{\partial D} Mdx + Ndy &= \int_{C_1} \frac{xdy - ydx}{x^2+y^2} + \int_{C_h} \frac{xdy - ydx}{x^2+y^2} \\ &= \int_0^{2\pi} (\cos^2 t + \sin^2 t)dt + \int_{2\pi}^{0} \frac{h^2(\cos^2 t + \sin^2 t)}{h^2} dt \\ &= 2\pi - 2\pi = 0.\end{aligned}$$

Hence

$$\int_{\partial D} Mdx + Ndy = 0 = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy.$$

□

Example 15.4.5. Evaluate $\int_C \frac{xdy - ydx}{x^2+y^2}$ where C_* is any closed curve around the origin.

sol. Since the integrand is not continuous at $(0,0)$, we cannot use Green's theorem on the interior of C_*. But if we remove a small circle of radius h around the origin, we can use the Green's theorem on the region bounded by C_* and C_h (Fig **??**) as in the previous example to see

$$\int_{C_*} Mdx + Ndy = -\int_{C_h} Mdx + Ndy.$$

Now the integral $-\int_{C_h}(Mdx + Ndy)$ can be computed by polar coordinate: From

$$\begin{aligned} x &= h\cos\theta, \qquad y = h\sin\theta, \\ dx &= -h\sin\theta d\theta, \\ dy &= h\cos\theta d\theta, \end{aligned}$$

we see

$$\frac{xdy - ydx}{x^2+y^2} = \frac{h^2(\cos^2\theta + \sin^2\theta)}{h^2} d\theta = d\theta.$$

Hence

$$\int_{C_*} \frac{xdy - ydx}{x^2+y^2} = 2\pi.$$

□

Vector Form using the Curl

Any vector field in $\mathbb{R}^2$ can be treated as a vector field in $\mathbb{R}^3$. For example, the vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ on $\mathbb{R}^2$ can be viewed as $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + 0\mathbf{k}$. Then we can define its curl and it can be shown that the curl is (compute!) $(\partial N/\partial x - \partial M/\partial y)\mathbf{k}$. Then we obtain

$$(\mathrm{curl}\,\mathbf{F}) \cdot \mathbf{k} = \left[\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\mathbf{k}\right] \cdot \mathbf{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right).$$

Hence by Green's theorem,

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{x} = \int_{\partial D} M dx + N dy = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dxdy.$$

This is a vector form of Green's theorem.

Theorem 15.4.6. ***(Vector form of Green's theorem)*** *Let $D \subset \mathbb{R}^2$ be region with ∂D. If $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ is a $\mathcal{C}^1$-vector field on D then*

$$\int_{\partial D} \mathbf{F} \cdot d\mathbf{x} = \iint_D (\mathbf{curl}\ \mathbf{F}) \cdot \mathbf{k}\, dxdy = \iint_D (\nabla \times \mathbf{F}) \cdot \mathbf{k}\, dxdy.$$

15.5 (Parameterized) Surfaces and Surface area

Definition 15.5.1. A parameterized surface is a (one-to-one) function $\mathbf{r}\colon D \subset \mathbb{R}^2 \to \mathbb{R}^3$

$$\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)).$$

Normal Vectors, Tangent Planes, and Surface Area

First look at the case when the surface is the graph of $f\colon D \to \mathbb{R}$. Then we have

$$\mathbf{r}(x, y) = (x, y, f(x, y)).$$

First fix $y = y_0$ and then $x = x_0$. The derivatives of $\mathbf{r}$ in the direction of x-axis and y-axis at $\mathbf{r}(x_0, y_0) = (x_0, y_0, f(x_0, y_0))$ are

$$\mathbf{r}_x(x_0, y_0) = \mathbf{i} + f_x(x_0, y_0)\mathbf{k}, \quad \mathbf{r}_y(x_0, y_0) = \mathbf{j} + f_y(x_0, y_0)\mathbf{k}.$$

These are nothing but the tangent vectors to the curves $\mathbf{r}(x, y_0)$ and $\mathbf{r}(x_0, y)$, respectively. Hence the normal vector is given by the cross product

$$\begin{aligned}
\mathbf{r}_x(x_0, y_0) \times \mathbf{r}_y(x_0, y_0) &= (\mathbf{i} + f_x(x_0, y_0)\mathbf{k}) \times (\mathbf{j} + f_y(x_0, y_0)\mathbf{k}) \\
&= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(x_0, y_0) \\ 0 & 1 & f_y(x_0, y_0) \end{vmatrix} \\
&= -f_x(x_0, y_0)\mathbf{i} - f_y(x_0, y_0)\mathbf{j} + \mathbf{k}.
\end{aligned}$$

In general, consider the surface parameterized by

$$\mathbf{r}(x(u, v), y(u, v)) = (x(u, v), y(u, v), z(u, v)).$$

Then we see two tangent vectors are

$$\begin{aligned}
\mathbf{r}_u &= \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\Big|_{(u_0, v_0)} \\
\mathbf{r}_v &= \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\Big|_{(u_0, v_0)}
\end{aligned}$$

These are obtained by considering the cross sections with the planes $v = v_0$ and $u = u_0$, respectively. If the normal vector

$$\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}$$

is nonzero, then we say the surface is **smooth**.

Definition 15.5.2. When $\mathbf{N}$ is a normal vector to a surface $\mathbf{r}$, the **tangent plane** at $\mathbf{r}(u_0, v_0) = (x_0, y_0, z_0)$ is defined by

$$\mathbf{N} \cdot (x - x_0, y - y_0, z - z_0) = 0.$$

Example 15.5.3. Consider the surface given by

$$x = u \cos v, \quad y = u \sin v, \quad z = u^2 + v^2.$$

Find the tangent plane at $\mathbf{r}(1, 0)$.

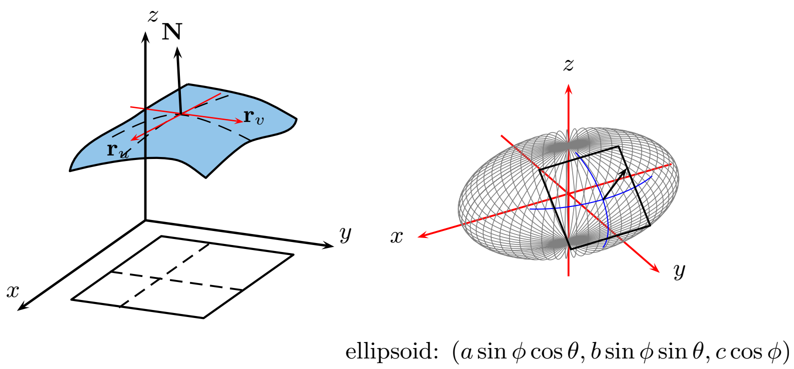


Figure 15.6: Coord. curves, Tangent vectors and normal vectors to a surface

sol. Since $\mathbf{r}(u,v) = (u\cos v, u\sin v, u^2 + v^2)$ we have

$$\mathbf{r}_v = (\cos v, \sin v, 2u), \quad \mathbf{r}_v = (-u\sin v, u\cos v, 2v).$$

Hence we see $\mathbf{r}_u \times \mathbf{r}_v = (-2u^2\cos v + 2v\sin v, -2u^2\sin v - 2v\cos v, u)$. Since $\mathbf{r}(1,0) = (1,0,1)$ and $\mathbf{N} = \mathbf{r}_u \times \mathbf{r}_v(1,0) = (-2,0,1)$, we see the tangent plane is given as

$$-2(x-1) + 0(y-0) + 1(z-1) = 0.$$

□

Area of Parameterized Surface

Recall 2-D case: When $\mathbf{r} : D \to R$ is a transformation in $\mathbb{R}^2$. Consider the small rectangle $A = [u, u+\Delta u] \times [v + \Delta v]$. The two tangent vectors $(\Delta u, 0)$ and $(0, \Delta v)$ are mapped to the boundary of image $\mathbf{r}(A)$ at $\mathbf{r}(u,v)$ as

$$\mathbf{r}_u \Delta u, \quad \mathbf{r}_v \Delta v.$$

These vectors form a parallelogram approximating the region $\mathbf{r}(A)$(figure **??**). The area of the parallelogram is

$$\begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v = \frac{\partial(x,y)}{\partial(u,v)} \Delta u \Delta v.$$

$$\|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u \Delta v = |J| \Delta u \Delta v.$$

Hence we have

$$\iint_R dxdy = \iint_D |J| dudv.$$

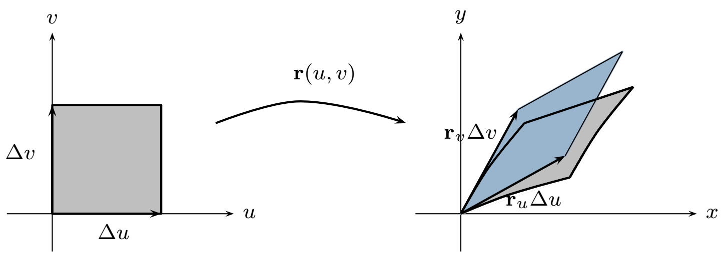


Figure 15.7: approximate $\mathbf{r}(A)$

Now we consider a surface lying in space: $\mathbf{r}\colon D \to \mathbb{R}^3$. Divide the domain D into small rectangles of the form $A = [u, u + \Delta u] \times [v, v + \Delta v]$. The image of A under $\mathbf{r}$ is a portion of the surface having four corners at

$$\mathbf{r}(u, v),\ \ \mathbf{r}(u + \Delta u, v),\ \ \mathbf{r}(u, v + \Delta v),\ \ \mathbf{r}(u + \Delta u, v + \Delta v).$$

This surface can be approximated by a parallelogram whose sides are given by(fig **??**) $\mathbf{r}_u(u, v)\Delta u$ and $\mathbf{r}_v(u, v)\Delta v$, where

$$\begin{aligned} \mathbf{r}_u &= \frac{\partial \mathbf{r}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k} \\ \mathbf{r}_v &= \frac{\partial \mathbf{r}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}. \end{aligned} \tag{15.8}$$

Hence the area of $\mathbf{r}(A)$ is (again like 2D) approximated by

$$\|\mathbf{r}_u \times \mathbf{r}_v\| \Delta u \Delta v.$$

Hence the area of the surface is the limit of sum of these.

Definition 15.5.4. We define the surface area $A(S)$ of a parameterized surface S by

$$A(S) = \iint_S dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| dudv.$$

We call $d\sigma = dS := \|\mathbf{r}_u \times \mathbf{r}_v\| dudv$ the **surface area differential**. Then

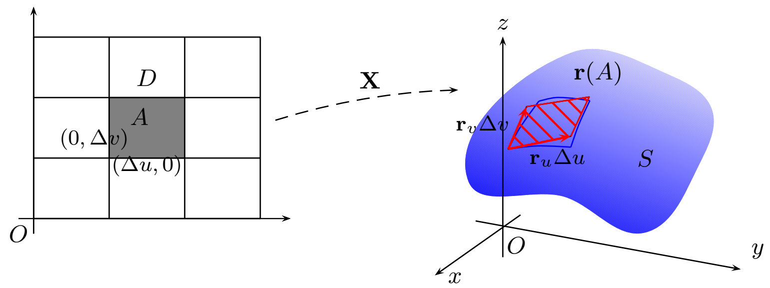


Figure 15.8: Approx. area of surface by a tangent plane

we see that [1]

$$\iint_{\mathbf{r}(D)} dS = \iint_D \|\mathbf{r}_u \times \mathbf{r}_v\| du dv.$$

Example 15.5.5 (Cone)**.** Let D be the surface of a cone given by

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = r, \ 0 \le r \le 1.$$

sol. Compute directly using $\|\mathbf{r}_r \times \mathbf{r}_\theta\| dr d\theta$. We see that $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = r\sqrt{2}$. Hence the area is

$$\begin{aligned}\iint_{\mathbf{r}(D)} dS &= \iint_D \|\mathbf{r}_r \times \mathbf{r}_\theta\| dr d\theta \\ &= \iint_D r\sqrt{2} dr d\theta \\ &= \int_0^{2\pi} \int_0^1 r\sqrt{2} dr d\theta = \pi\sqrt{2}.\end{aligned}$$

□

Example 15.5.6 (Football like surface)**.** Find the area of the surface of revolution of the curve $x = \cos z, y = 0, |z| \le \pi/2$ around z-axis.

[1] $\mathbf{r}$ is assumed to be 1-1.

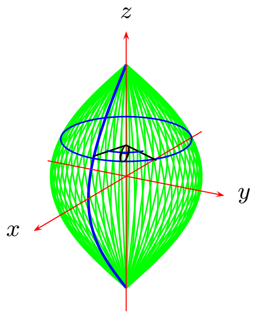
sol. The surface of revolution is parameterized by

$$\mathbf{r}(u,v) = (x,y,z),\ x = \cos u \cos v,\ y = \cos u \sin v,\ z = u, -\frac{\pi}{2} \le u \le \frac{\pi}{2},\ 0 \le v \le 2\pi.$$

We see

$$\begin{aligned}\mathbf{r}_u &= -\sin u \cos v \mathbf{i} - \sin u \sin v \mathbf{j} + \mathbf{k} \\ \mathbf{r}_v &= -\cos u \sin v \mathbf{i} + \cos u \cos v \mathbf{j}.\end{aligned}$$

Compute $\|\mathbf{r}_r \times \mathbf{r}_\theta\|$.



$$\begin{aligned}\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin u \cos v & -\sin u \sin v & 1 \\ -\cos u \sin v & \cos u \cos v & 0 \end{vmatrix} \\ &= -\cos u \cos v \mathbf{i} - \cos u \sin v \mathbf{j} - (\sin u \cos u)\mathbf{k} \\ \|\mathbf{r}_u \times \mathbf{r}_v\| &= \cos u \sqrt{1 + \sin^2 u}\end{aligned}$$

Hence the area is

$$\begin{aligned}A &= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} \cos u \sqrt{1+\sin^2 u}\, du dv \\ &= 2\int_0^{2\pi} \int_0^{\pi/2} \sqrt{1+t^2}\, dt dv (\text{ need table}) \\ &= \int_0^{2\pi} \left[t\sqrt{1+t^2} + \ln(t + \sqrt{1+t^2}) \right]_0^1 dv \\ &= 2\pi \left[\sqrt{2} + \ln(1+\sqrt{2}) \right].\end{aligned}$$

□

Implicit Surfaces

Assume a surface is defined implicitly by

$$F(x,y,z) = c.$$

In this case, it is not easy to find the explicit form of parametrization. However, we can still compute

$$dS = \left\| \left(\frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}\right) \times \left(\frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}\right) \right\| dudv \tag{15.9}$$

from the implicit expression. Assume the surface is defined over a region R having $\mathbf{k}$ as the unit normal vector. Define the parameters $x = u, y = v$ then $z(x, y) = z(u, v)$.

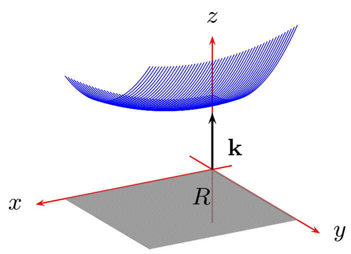


Figure 15.9: Implicit surface $F(x, y, z) = c$ with normal vector $\mathbf{k}$ on R

Assume the surface has the following parametrization

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + h(u, v)\mathbf{k}. \tag{15.10}$$

Then

$$\mathbf{r}_u = \mathbf{i} + \frac{\partial h}{\partial u}\mathbf{k} \text{ and } \mathbf{r}_v = \mathbf{j} + \frac{\partial h}{\partial v}\mathbf{k}. \tag{15.11}$$

Taking derivative w.r.t x (and y resp.) using implicit differentiation, we get

$$F_x + \frac{\partial z}{\partial x} = 0 \text{ and } F_y + \frac{\partial z}{\partial y} = 0.$$

From this we get

$$\frac{\partial h}{\partial u} = -\frac{F_x}{F_z} \text{ and } \frac{\partial h}{\partial v} = -\frac{F_y}{F_z}.$$

Hence

$$\mathbf{r}_u = \mathbf{i} - \frac{F_x}{F_z}\mathbf{k} \text{ and } \mathbf{r}_v = -\frac{F_y}{F_z}\mathbf{k} \tag{15.12}$$

and

$$
\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \frac{F_x}{F_z}\mathbf{i} + \frac{F_y}{F_z}\mathbf{j} + \mathbf{k} \\
&= \frac{1}{F_z}(F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}) \\
&= \frac{\nabla F}{F_z} = \frac{\nabla F}{\nabla F \cdot \mathbf{k}}.
\end{aligned}
$$

The area of implicit surface $F(x, y, z) = c$ defined over R is

$$\iint_R \frac{|\nabla F|}{|\nabla F \cdot \mathbf{p}|} dA,$$

where $\mathbf{p} = \mathbf{i}, \mathbf{j}$ or $\mathbf{k}$ is the normal to R and $\nabla F \cdot \mathbf{p} \neq 0$.

Example 15.5.7. Find the area of surface of paraboloid $x^2 + y^2 - z = 0$ between $0 \le z \le 4$.

sol. Let $F(x, y, z) = x^2 + y^2 - z$ so that $\nabla F = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$. $\nabla F \cdot \mathbf{k} = -1$. With $D = \{x^2 + y^2 \le 4\}$, the area is

$$
\begin{aligned}
A &= \iint_D \sqrt{4x^2 + 4y^2 + 1}\, dxdy \\
&= \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r dr d\theta \\
&= \int_0^{2\pi} \frac{1}{12} \left[(4r^2 + 1)^{3/2}\right]_0^2 d\theta \\
&= \frac{\pi}{6}(17\sqrt{17} - 1).
\end{aligned}
$$

□

Surface Area of a Graph

When a surface S is given by the graph of function $z = f(x, y)$ on D, we see U is parameterized by $\mathbf{r}(x, y) = (x, y, f(x, y))$. Find $\mathbf{r}_x$, $\mathbf{r}_y$ by

$$\mathbf{r}_x = \mathbf{i} + f_x\mathbf{k}, \qquad \mathbf{r}_y = \mathbf{j} + f_y\mathbf{k}.$$

This corresponds to above case with $F(x, y, z) = z - f(x, y)$.

Since

$$\mathbf{r}_x \times \mathbf{r}_y = (\mathbf{i} + f_x\mathbf{k}) \times (\mathbf{j} + f_y\mathbf{k}) = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k},$$

the area is

$$\iint_{\mathbf{r}(D)} dS = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}dxdy.$$

Geometric interpretation

We refer to figure **??**. The unit normal vector $\mathbf{N}(x, y, z)$ on S is

$$\mathbf{N}(x, y, z) = -f_x\mathbf{i} - f_y\mathbf{j} + \mathbf{k}.$$

We can find the formula using the angle between $\mathbf{N}$ and $\mathbf{k}$. Let φ be the angle between $\mathbf{N}$ and $\mathbf{k}$. Then $\cos\varphi$ satisfies

$$\cos\varphi = \frac{\mathbf{N}\cdot\mathbf{k}}{\|\mathbf{N}\|} = \frac{1}{\sqrt{(f_x)^2 + (f_y)^2 + 1}}.$$

Hence

$$dS = \sqrt{(f_x)^2 + (f_y)^2 + 1}dxdy = \frac{dxdy}{\cos\varphi},$$

and we get

$$\iint_{\mathbf{r}} dS = \iint_D \frac{dxdy}{\cos\varphi}.$$

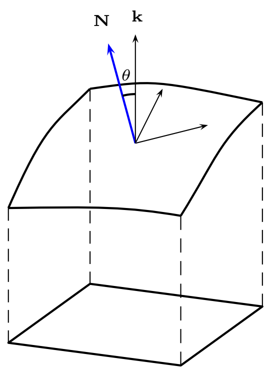


Figure 15.10: Ratio between two surface area is the cosine of angle

Example 15.5.8. Find the surface area of a unit ball.

sol. From $x^2 + y^2 + z^2 = 1$, we let $z = f(x, y) = \sqrt{1 - x^2 - y^2}$.

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{1 - x^2 - y^2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1 - x^2 - y^2}}.$$

Area of the half sphere is

$$\begin{aligned} \iint_S dS &= \iint_D \frac{1}{\sqrt{1 - x^2 - y^2}} dxdy \\ &= \int_0^{2\pi} \int_0^1 \frac{r}{\sqrt{1 - r^2}} drd\theta \\ &= 2\pi. \end{aligned}$$

□

Example 15.5.9. Let $\mathbf{r} = (r\cos\theta, r\sin\theta, \theta)$ be the parametrization of a helicoid-like surface S, where $0 \le r \le 1$, $0 \le \theta \le 2\pi$. Suppose S is covered with a metal of density m which equal to twice the distance to the central axis, i.e, $m = 2\sqrt{x^2 + y^2} = 2r$. Find the total mass of metal covering the surface.

sol. First we can show $\|\mathbf{r}_r \times \mathbf{r}_\theta\| = \sqrt{1 + r^2}$. Hence we have

$$\begin{aligned} M &= \iint_S 2r dS = 2 \iint_D r\|\mathbf{r}_r \times \mathbf{r}_\theta\| drd\theta \\ &= \int_0^{2\pi} \int_0^1 2r\sqrt{1 + r^2} drd\theta = \frac{4}{3}\pi(2^{3/2} - 1). \end{aligned}$$

□

15.6 Surface Integrals

Integrals of Scalar functions over Surface

Definition 15.6.1. Let S be a surface parameterized by $\mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v))$, where $(u, v) \in D$. Then the surface integral of a scalar function $f(x, y, z)$ defined on S is

$$\iint_S f\, dS = \iint_D f(x(u, v), y(u, v), z(u, v))\|\mathbf{r}_u \times \mathbf{r}_v\| dudv.$$

Surface integrals over graphs

Suppose S is the graph of a C^1 function $z = g(x, y)$. Then we parameterize it by

$$x = u, \quad y = v, \quad z = g(u, v)$$

and

$$\|\mathbf{r}_u \times \mathbf{r}_v\| = \sqrt{1 + (g_u)^2 + (g_v)^2}.$$

So the integral of f on S becomes

$$\iint_S f(x, y, z)\, dS = \iint_D f(x, y, g(x, y))\sqrt{1 + (g_x)^2 + (g_y)^2}\, dxdy.$$

Example 15.6.2. Evaluate $\iint_S z^2 dS$ when S is the unit sphere.

sol. The unit sphere is described by

$$\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \ (0 \le \theta \le 2\pi, \ 0 \le \phi \le \pi).$$

Since

$$\|\mathbf{r}_\phi \times \mathbf{r}_\theta\| = \sin\phi$$

and $z^2 = \cos^2\phi$, we have

$$\begin{aligned}\iint_S z^2 dS &= \iint_D \cos^2\phi \|\mathbf{r}_\theta \times \mathbf{r}_\phi\| d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} \cos^2\phi\sin\phi\, d\phi d\theta \\ &= \frac{4\pi}{3}.\end{aligned}$$

□

Example 15.6.3. Evaluate $\iint_S G(x, y, z)dS$ over a football like surface S

$$x = \cos u\cos v, \ y = \cos u\sin v, \ z = u, -\frac{\pi}{2} \le u \le \frac{\pi}{2}, 0 \le v \le 2\pi$$

when $G(x, y, z) = \sqrt{1 - x^2 - y^2}$.

sol. Over the football surface the function G is given by

$$\sqrt{1 - x^2 - y^2} = \sqrt{1 - \cos^2 u} = |\sin u|.$$

The surface differential is (Ref. Example **??**)

$$dS = \cos u \sqrt{1 + \sin^2 u}\, du dv.$$

Hence

$$\begin{aligned}
\iint_S \sqrt{1 - x^2 - y^2} dS &= \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} |\sin u| \cos u \sqrt{1 + \sin^2 u}\, du dv \\
&= 2 \int_0^{2\pi} \int_{-\pi/2}^{\pi/2} |\sin u| \cos u \sqrt{1 + \sin^2 u}\, du dv \\
&= \int_0^{2\pi} \int_1^2 \sqrt{w} dw dv \\
&= 2\pi \cdot \frac{2}{3} w^{3/2}|_1^2 = \frac{4\pi}{3}(2\sqrt{2} - 1).
\end{aligned}$$

□

Example 15.6.4. Evaluate $\iint_S \sqrt{x(1+2z)} dS$ where $S = \{z = y^2/2, x, y \geq 0,\ x + y \leq 1\}$.

sol. This is an integral over a graph of a function. Let $z = g(x, y) = y^2/2$ so that the surface differential is

$$dS = \sqrt{g_x^2 + g_y^2 + 1} dx dy = \sqrt{y^2 + 1} dx dy.$$

The surface area is

$$\begin{aligned}
\iint_S \sqrt{x(1+2z)} \sqrt{y^2 + 1} dx dy &= \int_0^1 \int_0^{1-x} \sqrt{x}(y^2 + 1) dy dx \\
&= \int_0^1 \sqrt{x}((1 - x) + \frac{1}{3}(1 - x)^3) dx.
\end{aligned}$$

□

Orientation

Let $\mathbf{r} \colon D \to \mathbb{R}^3$ represent an oriented surface. If $\mathbf{n}(\mathbf{r})$ is the unit normal to S, then

$$\mathbf{n}(\mathbf{r}) = \pm \frac{\mathbf{r}_u \times \mathbf{r}_v}{\|\mathbf{r}_u \times \mathbf{r}_v\|}.$$

We choose a parametrization so that the sign is positive (**orientation-preserving**)

Surfaces Integrals of vector Fields

Definition 15.6.5. The surface integral of $\mathbf{F}$ on a surface S is the surface integral of normal projection of $\mathbf{F}$ to the surface S.

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

If $\mathbf{F}$ represents the velocity of a fluid, then the surface integral is the **amount of fluid that passes through the surface (per unit time)**.

Since $\mathbf{n} = \mathbf{r}_u \times \mathbf{r}_v / \|\mathbf{r}_u \times \mathbf{r}_v\|$ is the unit normal vector to the surface,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_D \mathbf{F}\cdot\mathbf{n}\|\mathbf{r}_u\times\mathbf{r}_v\|dudv\\ &= \iint_D \mathbf{F}\cdot\frac{\mathbf{r}_u\times\mathbf{r}_v}{\|\mathbf{r}_u\times\mathbf{r}_v\|}\|\mathbf{r}_u\times\mathbf{r}_v\|\,dudv\\ &:= \iint_{\mathbf{r}(D)} \mathbf{F}\cdot d\mathbf{S}.\end{aligned}$$

Example 15.6.6. Find the flux of $\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}$ through the surface S given by

$$y = x^2,\ 0 \le x \le 1,\ 0 \le z \le 4.$$

sol. We can parameterize the surface using (x, z). $\mathbf{r} = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}$. So

$$\begin{aligned}\mathbf{r}_x &= \mathbf{i} - 2x\mathbf{j}, \quad \mathbf{r}_z = \mathbf{k}\\ \mathbf{r}_x \times \mathbf{r}_z &= 2x\mathbf{i} - \mathbf{j}\\ \mathbf{n} &= \frac{2x\mathbf{i} - \mathbf{j}}{\sqrt{4x^2+1}}.\end{aligned}$$

On the surface

$$\mathbf{F} = yz\mathbf{i} + x\mathbf{j} - z^2\mathbf{k} = x^2z\mathbf{i} + x\mathbf{j} - z^2\mathbf{k}.$$

Hence

$$\begin{aligned}
\mathbf{F}\cdot\mathbf{n} &= \frac{1}{\sqrt{4x^2+1}}(x^2z\cdot 2x - x)\\
&= \frac{2x^3z-x}{\sqrt{4x^2+1}},\\
\iint_S \mathbf{F}\cdot\mathbf{n}dS &= \int_0^4\int_0^1 \frac{2x^3z-x}{\sqrt{4x^2+1}}\|\mathbf{r}_x\times\mathbf{r}_z\|dxdz\\
&= \int_0^4\int_0^1 (2x^3z-x)xdz = 2.
\end{aligned}$$

□

Example 15.6.7. Let S be the unit sphere parameterized by

$$\mathbf{r}(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),\ (0\le\theta\le 2\pi,\ 0\le\phi\le\pi).$$

Compute $\iint_S \mathbf{r}\cdot d\mathbf{S}$, where $\mathbf{r} = x\mathbf{i}+y\mathbf{i}+z\mathbf{k}$ denotes the position vector.

sol. We see

$$\begin{aligned}
\mathbf{r}_\phi &= \cos\phi\cos\theta\mathbf{i} + \cos\phi\sin\theta\mathbf{j} - \sin\phi\mathbf{k},\\
\mathbf{r}_\theta &= -\sin\phi\sin\theta\mathbf{i} + \sin\phi\cos\theta\mathbf{j},\\
\mathbf{r}_\phi\times\mathbf{r}_\theta &= \sin\phi(\cos\theta\sin\phi\mathbf{i} + \sin\theta\sin\phi\mathbf{j} + \cos\phi\mathbf{k}).
\end{aligned}$$

Hence $\mathbf{r}\cdot d\mathbf{S} = \mathbf{r}\cdot(\mathbf{r}_\phi\times\mathbf{r}_\theta)d\phi\, d\theta = \sin\phi\, d\phi\, d\theta$ and

$$\iint_S \mathbf{r}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi}\sin\phi\, d\phi\, d\theta = 4\pi.$$

□

Surface Integral of vector fields over Graphs

Suppose S is the graph of $z = g(x,y)$. We parameterize the surface S by $\mathbf{r}(x,y) = (x,y,g(x,y))$ and compute

$$\mathbf{r}_x = \mathbf{i} + g_x\mathbf{k},\quad \mathbf{r}_y = \mathbf{j} + g_y\mathbf{k}.$$

Hence

$$\mathbf{r}_x\times\mathbf{r}_y = -(g_x)\mathbf{i} - (g_y)\mathbf{j} + \mathbf{k}$$

and we see

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) dxdy = \iint_D [F_1(-g_x) + F_1(-g_y) + F_3]\, dxdy.$$

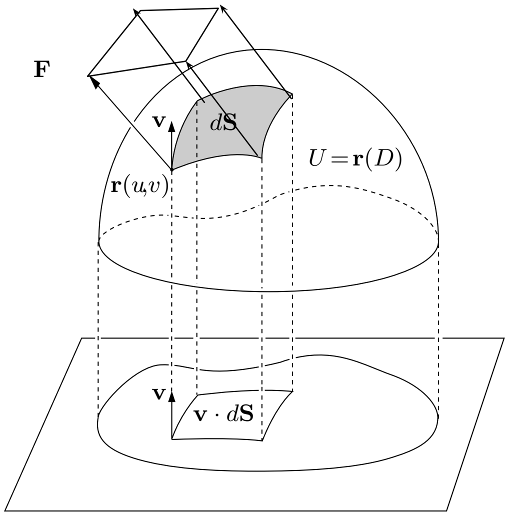


Figure 15.11: Area of shadow region and flux across S

Example 15.6.8 (Gauss Law)**.** The flux of an electric field $\mathbf{E}$ over a closed surface S is the net charge Q contained in the surface. Namely,

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = Q.$$

Suppose $\mathbf{E} = E\mathbf{n}$(constant multiple of the unit normal vector) then

$$\iint_S \mathbf{E} \cdot d\mathbf{S} = \iint_S E dS = Q = E \cdot A(S).$$

So $E = \frac{Q}{A(S)}$ and if S is sphere of radius R then

$$E = \frac{Q}{4\pi R^2}. \tag{15.13}$$

Example 15.6.9. Given a disk lying on the plane $z = 12$ described by

$$z = 12, \quad x^2 + y^2 \leq 25,$$

compute $\iint_S \mathbf{r} \cdot d\mathbf{S}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

sol. We see

$$\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} \times \mathbf{j} = \mathbf{k}.$$

So $\mathbf{r} \cdot (\mathbf{r}_x \times \mathbf{r}_y) = z$ and

$$\iint_S \mathbf{r} \cdot d\mathbf{S} = \iint_D z dx dy = 12A(D) = 300\pi.$$

□

Summary

(1) Given a parameterized surface $\mathbf{r}(u, v)$

(a) Surface integral of a scalar function f:

$$\iint_{\mathbf{r}(D)} f dS = \iint_D f(\mathbf{r}(u, v)) \|\mathbf{r}_u \times \mathbf{r}_v\| du dv$$

(b) Scalar surface element:

$$dS = \|\mathbf{r}_u \times \mathbf{r}_v\| du dv$$

(c) Integral of a vector field:

$$\iint_{\mathbf{r}(D)} \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du dv = \iint_S (\mathbf{F} \cdot \mathbf{n})\, dS$$

(d) Vector surface element:

$$d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)\, du dv = \mathbf{n}\, dS$$

(2) When the surface is given by a graph $z = g(x, y)$

(a) Integral of a scalar f:

$$\iint_S f dS = \iint_D f(x, y, g(x, y)) \sqrt{(g_x)^2 + (g_y)^2 + 1}\, dx dy$$

(b) Scalar surface element:

$$dS = \frac{dx\,dy}{\cos\theta} = \sqrt{(g_x)^2 + (g_y)^2 + 1}\;dxdy$$

(c) Integral of a vector field:

$$\iint_{\mathbf{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_D (-F_1 g_x - F_2 g_y + F_3)\,dxdy$$

(d) Vector surface element:

$$d\mathbf{S} = \mathbf{n}\,dS = (-g_x\mathbf{i} - g_y\mathbf{j} + \mathbf{k})\,dxdy$$

15.7 Stokes' Theorem

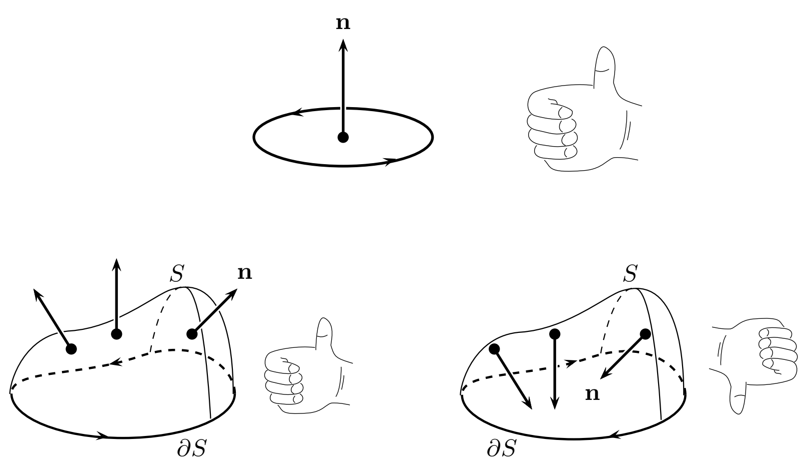


Figure 15.12: Orientation by right handed rule

Stokes' theorem is the generalization of Green's theorem to the surface lying in $\mathbb{R}^3$: Consider a simple closed curve lying in $\mathbb{R}^3$ and a surface having the curve as boundary: Caution: there are many surfaces having the same curve as boundary. But as long as the vector fields are C^1 in a large region containing the curve and the surface, any surface play the same role.

Recall : the curl of $\mathbf{F}$: $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, then

$$\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F} = \Big(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\Big)\mathbf{k}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}.$$

Theorem 15.7.1 (Stokes' theorem)**.** *Let S be a piecewise smooth oriented surface. Suppose the boundary ∂S consists of finitely many piecewise C^1 curve with the same orientation with S. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a C^1-vector field defined on S. Then*

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

For a 2D surface this reduces to the Green's Theorem:

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{k} dA = \iint_S \Big(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\Big) dxdy = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{r}.$$

Corollary 15.7.2. *If S_1 and S_2 are two surfaces having the same boundary, then*

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = \iint_{S_2} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

Example 15.7.3. Let S be smooth surface having an oriented simple closed curve C as boundary and let $\mathbf{F} = ye^z\mathbf{i} + xe^z\mathbf{j} + xye^z\mathbf{k}$. Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$.

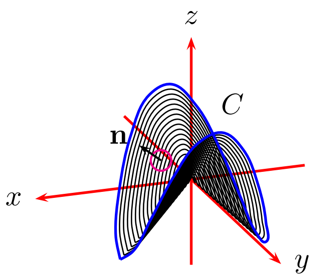
$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & xe^z & xye^z \end{vmatrix} = 0.$$

By Stoke's theorem,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 0.$$

Example 15.7.4. Calculate the circulation of $\mathbf{F} = (x^2 - y)\mathbf{i} + 4z\mathbf{j} + x^2\mathbf{k}$ around the circle C where the plane $z = 2$ meets the cone $z = \sqrt{x^2 + y^2}$, counterclockwise. (In two ways)

sol. One way is to directly compute the circulation (Easy, skip it). But



Surface $z = y^2 - x^2, x^2 + y^2 \leq 1$ for Example **??**

another way is to use Stokes' theorem on the given surface. This make things worse!!! (see book Example 4, p. 1019)

However, we can use a flat disc $z = 2$ having the same curve C as the boundary. On that disc $\mathbf{n} = \mathbf{k}$ and $\nabla \times \mathbf{F} = -4\mathbf{i} - 2x\mathbf{j} + \mathbf{k}$. $\nabla \times \mathbf{F} \cdot \mathbf{n} = 1$. So by Stokes theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} dS \quad = \quad \iint_{x^2+y^2 \leq 4} 1 dA = 4\pi.$$

□

Example 15.7.5. Consider a surface S formed by hyperbolic paraboloid $z = y^2 - x^2$ lying inside the cylinder of radius one around z axis and the boundary curve C. (Fig **??**) Compute the circulation of $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$ around C.(assume normal vector has positive $\mathbf{k}$ component on S)

sol. First we find the boundary curve C. Since it is intersection with cylinder $r = 1$, we can use

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + (\sin^2 t - \cos^2 t)\mathbf{k}$$

We calculate the circulation of $\mathbf{F} = y\mathbf{i} - x\mathbf{j} + x^2\mathbf{k}$ around the boundary curve C.

$$\frac{d\mathbf{r}}{dt} = -\sin t\mathbf{i} + \cos t\mathbf{j} + (4 \sin t \cos t)\mathbf{k}$$

and on the curve $\mathbf{r}$ the vector field is

$$\mathbf{F} = \sin t\mathbf{i} - \cos t\mathbf{j} + \cos^2 t\mathbf{k}$$

$$\begin{aligned}\int_0^{2\pi} \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}dt &= \int_0^{2\pi}\left(-\sin^2 t-\cos^2 t+4\sin t\cos^3 t\right)dt\\ &= \int_0^{2\pi}\left(4\sin t\cos^3 t-1\right)dt=-2\pi\end{aligned}$$

However, the use of Stokes' theorem for this problem make it worse, terrible!!!

□

Example 15.7.6. Verify Stokes' theorem when $\mathbf{F}=(x^2+y)\mathbf{i}+(x^2+2y)\mathbf{j}+2z^3\mathbf{k}$ and $C: x^2+y^2=4,\ z=2$.

sol. Show that $\int_C \mathbf{F}\cdot d\mathbf{s}=-4\pi$(easy). Let S be the disk $\{(x,y,z): x^2+y^2=4, z=2\}$. If $\mathbf{n}$ is the unit normal to S, then $\mathbf{n}=\mathbf{k}$ and

$$\begin{aligned}\nabla\times\mathbf{F} &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+y & x^2+2y & 2z^3\end{vmatrix}\\ &= (0-0)\mathbf{i}-(0-0)\mathbf{j}+(2x-1)\mathbf{k}=(2x-1)\mathbf{k}.\end{aligned}$$

Hence

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{s} &= \iint_S(\nabla\times\mathbf{F})\cdot d\mathbf{S}=\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}dS\\ &= \iint_S(2x-1)\mathbf{k}\cdot\mathbf{k}dS=\int_{-2}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}(2x-1)dxdy\\ &= -2\int_{-2}^{2}\sqrt{4-y^2}dy=-4\pi.\end{aligned}$$

□

Example 15.7.7. Evaluate

$$\int_C -y^3dx+x^3dy-z^3dz$$

where C is the intersection of the cylinder $x^2+y^2=1$ and plane $x+y+z=1$.

sol. Let $\mathbf{F}=-y^3\mathbf{i}+x^3\mathbf{j}-z^3\mathbf{k}$. Then above integral is $\int_C \mathbf{F}\cdot d\mathbf{r}$. If we consider any reasonable surface S having C as boundary, we can use Stokes' theorem with $\operatorname{curl}\mathbf{F} = 3(x^2+y^2)\mathbf{k}$. Let us assume S is the surface defined by $x+y+z=1$, $x^2+y^2 \le 1$. A parametrization of S is given by $\mathbf{r} = (u, v, 1-u-v)$. We need to compute

$$d\mathbf{S} = (\mathbf{r}_u \times \mathbf{r}_v)dudv = ((\mathbf{i}-\mathbf{k}) \times (\mathbf{j}-\mathbf{k}) = \mathbf{i}+\mathbf{j}+\mathbf{k})dudv.$$

Hence

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D 3(x^2+y^2)dxdy = \frac{3\pi}{2}.$$

Here the domain D is the set $\{(x,y)|x^2+y^2 \le 1\}$.

□

Example 15.7.8. A surface S is defined by $z = e^{-(x^2+y^2)}$ for $z \ge 1/e$. Let

$$\mathbf{F} = (e^{y+z} - 2y)\mathbf{i} + (xe^{y+z} + y)\mathbf{j} + e^{x+y}\mathbf{k}.$$

Evaluate $\iint_S \nabla \times \mathbf{F}\cdot d\mathbf{S}$.

sol. We see

$$\nabla \times \mathbf{F} = (e^{x+y} - xe^{y+z})\mathbf{i} + (e^{y+z} - e^{x+y})\mathbf{j} + 2\mathbf{k}$$

and

$$\mathbf{N} = 2xe^{-(x^2+y^2)}\mathbf{i} + 2ye^{-(x^2+y^2)}\mathbf{j} + \mathbf{k}.$$

So direct computation of $\int_S \nabla \times \mathbf{F}\cdot d\mathbf{S}$ seems almost impossible. Now try to use Stoke's theorem. First parameterize the boundary by

$$x = \cos t, y = \sin t, z = 1/e.$$

Then

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (e^{\sin t + 1/e} - 2\sin t, \cdots, e^{\cos t + \sin t}) \cdot (-\sin t, \cos t, 0)\, dt$$

This again is very difficult! Now think of another way. Think of another surface S' which has the same boundary as S., i.e, let S' be the unit disk $x^2+y^2 \le 1, z = 1/e$. Then $\mathbf{n} = \mathbf{k}$ and hence

$$\iint_S \nabla \times \mathbf{F}\cdot d\mathbf{S} = \iint_{S'} \nabla \times \mathbf{F}\cdot \mathbf{n}dS = \iint_{S'} 2dS = 2\pi.$$

□

Curl as Circulation - Paddle Wheel interpretation

By Stokes' theorem,

$$\int_{\partial S_\rho} \mathbf{F} \cdot d\mathbf{r} = \iint_{S_\rho} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS. \tag{15.14}$$

Hence dividing equation (**??**) we see

$$\begin{aligned} \lim_{\rho \to 0} \frac{1}{\pi\rho^2} \int_{\partial S_\rho} \mathbf{F} \cdot d\mathbf{s} &= \lim_{\rho \to 0} \frac{1}{\pi\rho^2} \iint_{S_\rho} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS \\ &= \lim_{\rho \to 0} (\nabla \times \mathbf{F}(Q)) \cdot \mathbf{n}(Q) \\ &= (\nabla \times \mathbf{F}) \cdot \mathbf{n}|_P. \end{aligned}$$

Thus curl of a vector field measures the circulation.

15.8 Divergence Theorem

We define the divergence of a vector field $\mathbf{F}$ as

$$\operatorname{div} \mathbf{F} = \nabla \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Physical meaning of divergence: Expansion or compression of a material.

Theorem 15.8.1. *[**Gauss' Divergence Theorem**] Let Ω be an elementary region in $\mathbb{R}^3$ and $\partial\Omega$ consists of finitely many oriented piecewise smooth closed surfaces. Let $\mathbf{F}$ be a $\mathcal{C}^1$ vector field on a region containing Ω. Then*

$$\iint_{\partial\Omega} \mathbf{F} \cdot \mathbf{n} dS = \iiint_{\Omega} \operatorname{div} \mathbf{F} dV.$$

The flux of a vector field $\mathbf{F}$ across Ω is equal to the integral of $\operatorname{div} \mathbf{F}$ in Ω.

Example 15.8.2. S is the unit sphere $x^2+y^2+z^2 = 1$ and $\mathbf{F} = 2x\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$. Find $\iint_S \mathbf{F} \cdot \mathbf{n} dS$.

sol. Let Ω be the region inside S. By Gauss theorem, it holds that

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iiint_{\Omega} \operatorname{div} \mathbf{F} dV.$$

Since $\operatorname{div}\mathbf{F} = \nabla\cdot(2x\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) = 2(1+y+z)$, the rhs is

$$2\iiint_\Omega (1+y+z)dV = 2\iiint_\Omega 1dV + 2\iiint_\Omega ydV + 2\iiint_\Omega zdV.$$

By symmetry, we have

$$\iiint_\Omega ydV = \iiint_\Omega zdV = 0.$$

Hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}dS = 2\iiint_\Omega (1+y+z)dV = 2\iiint_\Omega 1dV = \frac{8}{3}\pi.$$

□

Example 15.8.3. Find the flux of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ through the box cut from the first octant by the planes $x = 1, y = 1, z = 1$.

sol. Let Ω be the region inside S. By Gauss theorem, it holds that

$$\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iiint_\Omega \operatorname{div}\mathbf{F}dV.$$

Since $\operatorname{div}\mathbf{F} = \nabla\cdot(xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}) = x + y + z$, the rhs is

$$\iiint_\Omega (x+y+z)dV = \int_0^1\int_0^1\int_0^1 (x+y+z)dxdydz = \frac{3}{2}.$$

□

Theorem 15.8.4. *[**Divergence of curl**] Let* $\mathbf{F}$ *be a* $\mathcal{C}^2$ *vector field defined on a region containing* Ω. *Then*

$$div(\mathrm{curl}\mathbf{F}) = \nabla\cdot(\nabla\times\mathbf{F}) = 0.$$

Example 15.8.5. Show Gauss' theorem holds for $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ in $\Omega : x^2 + y^2 + z^2 \le a^2$.

sol. First compute $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F}$,

$$\operatorname{div}\mathbf{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

So

$$\iiint_\Omega (\operatorname{div} \mathbf{F}) dV = \iiint_\Omega 3\, dV = 3\Big(\frac{4}{3}\pi a^3\Big) = 4\pi a^3.$$

To compute the surface integral, we need to find the unit normal $\mathbf{n}$ on $\partial\Omega$. Since $\partial\Omega$ is the level set of $f(x,y,z) = x^2+y^2+z^2-a^2$, we see the unit normal vector to $\partial\Omega$ is

$$\mathbf{n} = \frac{\nabla f}{||\nabla f||} = \frac{2(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})}{\sqrt{4(x^2+y^2+z^2)}} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{a}.$$

So when $(x,y,z) \in \partial\Omega$,

$$\mathbf{F}\cdot\mathbf{n} = \frac{x^2+y^2+z^2}{a} = \frac{a^2}{a} = a$$

and

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n} dS = \iint_{\partial\Omega} a\, dS = a(4\pi a^2) = 4\pi a^3.$$

Hence

$$\iiint_\Omega (\operatorname{div} \mathbf{F}) dV = 4\pi a^3 = \iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n} dS.$$

and Gauss' theorem holds.

□

Example 15.8.6. Let Ω be the region given by $x^2+y^2+z^2 \le 1$. Find $\iint_{\partial\Omega}(x^2+4y-5z)dS$ by Gauss' theorem.

sol. To use Gauss' theorem, we need a vector field $\mathbf{F} = F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$ such that $\mathbf{F}\cdot\mathbf{n} = x^2+4y-5z$. Since the unit normal vector is $\mathbf{n} = x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, one such obvious choice is $\mathbf{F} = x\mathbf{i}+4\mathbf{j}-5\mathbf{k}$. Hence we have $\operatorname{div}\mathbf{F} = 1+0+(-0) = 1$. Now by Gauss theorem

$$\begin{aligned}\iint_{\partial\Omega}(x^2+4y-5z)dS &= \iint_{\partial\Omega}(x\mathbf{i}+4\mathbf{j}-5\mathbf{k})\cdot\mathbf{n}dS\\ &= \iint_{\partial\Omega}\mathbf{F}\cdot\mathbf{n}dS = \iiint_\Omega \operatorname{div}\mathbf{F}dV\\ &= \iiint_\Omega 1\ dV = \frac{4}{3}\pi.\end{aligned}$$

□

Example 15.8.7. Let Ω be the region satisfying $0 < b^2 \le x^2+y^2+z^2 \le a^2$. Find the flux of the vector field $\mathbf{F} = (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})/\rho^3$, $\rho = \sqrt{x^2+y^2+z^2}$

across the boundary of Ω.

sol. On the boundary of Ω, $\mathbf{n} = \pm(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\rho$. Hence $\mathbf{F}\cdot\mathbf{n} = \pm(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$,

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_a} \mathbf{F}\cdot\mathbf{n}dS - \iint_{S_a} \mathbf{F}\cdot\mathbf{n}dS$$

$$\iint_{S_a} \mathbf{F}\cdot\mathbf{n}dS = \iint_{\rho=a} \frac{1}{\rho^2}dS = 4\pi$$

Thus

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}dS = 4\pi - 4\pi = 0.$$

To use Gauss' theorem, we compute that $\nabla\cdot\mathbf{F} = 0$. Hence Now by Gauss theorem

$$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}dS = \iiint_{\Omega} \operatorname{div}\mathbf{F}dV = 0.$$

□

Divergence as flux per unit Volume

As we have seen before that $\operatorname{div}\mathbf{F}(P)$ is the *rate of change of total flux* at P per unite volume. Let Ω_ρ be a ball of radius ρ center at P. Then for some Q in Ω_ρ,

$$\iint_{\partial\Omega_\rho} \mathbf{F}\cdot\mathbf{n}dS = \iiint_{\Omega_\rho} \operatorname{div}\mathbf{F}dV = \operatorname{div}\mathbf{F}(Q)\cdot\operatorname{Vol}(\Omega_\rho).$$

Dividing by the volume we get

$$\operatorname{div}\mathbf{F}(Q) = \frac{1}{\operatorname{Vol}(\Omega_\rho)}\iint_{\partial\Omega_\rho} \mathbf{F}\cdot\mathbf{n}dS. \tag{15.15}$$

Taking the limit, we see

$$\lim_{\rho\to 0}\frac{1}{\operatorname{Vol}(\Omega_\rho)}\iint_{\partial\Omega_\rho} \mathbf{F}\cdot\mathbf{n}dS = \operatorname{div}\mathbf{F}(P). \tag{15.16}$$

Now we can give a physical interpretation: If $\mathbf{F}$ is the velocity of a fluid, then

$\operatorname{div}\mathbf{F}(P)$ is the rate at which the fluid flows out per unit volume.

Example 15.8.8. Find $\iint_S \mathbf{f}\cdot d\mathbf{S}$, where $\mathbf{F} = xy^2\mathbf{i} + x^2y\mathbf{j} + y\mathbf{k}$ and S is the surface of the the cylindrical region $x^2 + y^2 = 1$ bounded by the planes $z = 1$ and $z = -1$.

sol. Let W denote the solid region given above. By divergence theorem,

$$\begin{aligned}\iiint_W \operatorname{div}\mathbf{F}\,dV &= \iiint_W (x^2+y^2)dxdydz\\ &= \int_{-1}^{1}\left(\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy\right)dz\\ &= 2\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy.\end{aligned}$$

Now by polar coordinate,

$$2\iint_{x^2+y^2\le 1}(x^2+y^2)dxdy = 2\int_0^{2\pi}\int_0^1 r^3 dr d\theta = \pi.$$

□

Gauss' Law

Now apply Gauss' theorem to a region with a hole and get an important result in physics:

The electric field created by a point charge q at the origin is

$$\mathbf{E}(x,y,z) = \frac{q}{4\pi\epsilon_0}\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{r^3} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3},\ r = \sqrt{x^2+y^2+z^2}$$

Theorem 15.8.9. ***(Gauss' Law)*** *Let M be a region in $\mathbb{R}^3$ and $O \notin \partial M$. Then*

$$\iint_{\partial M}\mathbf{E}\cdot\mathbf{n}dS = \frac{q}{4\pi\epsilon_0}\iint_{\partial M}\frac{\mathbf{r}\cdot\mathbf{n}}{r^3}dS = \begin{cases}0 & \text{if } O\notin M\ ,\\ \frac{q}{\epsilon_0} & \text{if } O\in M.\end{cases}$$

Several versions of Green's theorem:

Tangential form	$\oint_C \mathbf{F}\cdot\mathbf{T}ds$	$=$	$\iint_R \nabla\times\mathbf{F}\cdot\mathbf{k}dA$
Stokes' theorem	$\oint_{\partial S} \mathbf{F}\cdot\mathbf{T}ds$	$=$	$\iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}dS$
Normal form	$\oint_C \mathbf{F}\cdot\mathbf{n}ds$	$=$	$\iint_R \nabla\cdot\mathbf{F}dA$
Divergcenc theorem	$\iint_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}dS$	$=$	$\iiint_\Omega \nabla\cdot\mathbf{F}dV$